Lecture group on algebraic K-theory

Session 1 (01/08/2025)

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Contents

Introduction

Every rings in these article will be commutative... Though I'm not quite sure for $GL_n(A)$.

The following theorem classify finitely generated modules on a principal ideal domain.

Theoreme 0.1 (Finitely generated modules over a PID)**.** *Let A be a PID and M a finitely generated module. There exists $s \in \mathbb{N}$ and $d_1|d_2|...|d_s$ non invertible elements of A such that $M \simeq A/(d_1) \times ... \times A/(d_s)$.*
Moreover s and the d_i are unique (up to association).

Thus on a PID, any module can be decomposed into a free part A^d and a torsion part $A/a_1A \times \cdots \times A/a_nA$ with the $a_n \neq 0$.

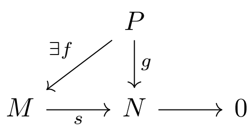
What if A is not a PID ? For example, if $A = \mathbb{Z}[i\sqrt{5}]$ and M is the idal $(2, 1 + i\sqrt{5})$, since M is not principal, it's not free, but it is torsion free. That's why we introduce the notion of projective module which allows us to treat the case of some usual torsion free modules.

1 Projective modules

This first section is mainly inspired from [2].

Definition 1.1 (Projective modules)**.** *A A-module P is projective if it satisfies one of the following equivalent conditions :*

1. *There exists a A-module Q such that $P \oplus Q$ is free.*
2. *For every A-modules M and N, every surjective morphism $s : M \to N$ and every morphism $g : P \to N$, there is a morphism $f : P \to M$ such that $g = s \circ f$.*



3. *Any exact sequence of the form $0 \to M \to N \to P \to 0$ splits.*

Proof. **1⇒2.** We extend g to $\tilde{g} : P \oplus Q \to N$ by $\tilde{g}(x, y) = g(x)$. As $P \oplus Q$ is free, we can take a basis $(e_i)_{i \in I}$ and define $\tilde{f} : P \oplus Q \to M$ by $\tilde{f}(e_i) = y_i$ where the y_i satisfy $s(y_i) = \tilde{g}(e_i)$. We then define f as the restriction of $\tilde{f}$ to P.

2⇒3. Let s be the morphism from N to P. Taking $g = Id$ in 2, we get a section $f : s \circ f = Id$ so we have $N \simeq P \oplus M$.

3⇒1. Let N the free R-module of basis $(e_x)_{x\in P}$ and s the surjection from N to P characterized by $\forall x \in P, s(e_x) = x$. The exact sequence $0 \to Ker(s) \to N \to P \to 0$ splits and gives $P \oplus Ker(s) = N$.

□

We can prove the following in the same way than the previous demonstration.

Proposition 1.2. *An A-module P is a finitely generated projective module if and only if there is an A-module Q and an integer $n \in \mathbb{N}$ such that $P \oplus Q \simeq A^n$.*

Let's give some examples and counterexamples : Donnons enfin quelques exemples de modules projectifs :

- Any sum / tensor product of projective modules is projective.
- Since on a PID, a submodule of a finitely generated free module is free. Thus every finitely generated projective module is free. Since fields are PID, it's also the case for fields.
- If A is integral, and I is an invertible fractionnal ideal of A, it is projective. Indeed, if $IJ = A$, there exists $x_1, \dots, x_n \in I$, and $y_1, \dots, y_n \in J$ such that $1 = x_1y_1 + \dots + x_ny_n$. Let define $u : \begin{cases} I \to A^n \\ x \mapsto (xy_1, ..., xy_n) \end{cases}$ and $v : \begin{cases} A^n \to I \\ (a_1, ..., a_n) \mapsto a_1x_1 + ... + a_nx_n \end{cases}$. Then $v \circ u = id_I$. This shows that I is isomorphic to a direct factor of A^n.

  In particular, any ideal of a Dedekind ring is projective. So it's the case of the example of the introduction.
- We will latter see than the converse of the last point is true : ideals that are projective are invertible. Then, since $(2, X)$ isn't an invertible ideal of $Z[X]$, it's not projective. More generally it's the case of any non principal ideal on an UFD.

The following example will allow us to study projective modules locally.

Proposition 1.3. *A finitely generated projective module on a local ring is free.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of A and let $P \oplus Q \simeq A^n$. Quotienting by $\mathfrak{m}$, we obtain $P/\mathfrak{m}P \oplus Q/\mathfrak{m}Q \simeq (A/\mathfrak{m})^n$. As $A/\mathfrak{m}$ is a Field, we have $P/\mathfrak{m}P \simeq (A/\mathfrak{m})^r$ and $Q/\mathfrak{m}Q \simeq (A/\mathfrak{m})^s$ with $n = r + s$. We can choose a basis $(f_i)_{1\le i\le n}$ of $(A/\mathfrak{m})^n$ such that the r first vectors give a basis of $P/\mathfrak{m}P$ and the last ones a basis of $Q/\mathfrak{m}Q$. Let's take $(e'_1, .., e'_r) \in P^r$ and $(e'_{r+1}, .., e'_n) \in Q^s$ such that e'_i reduces to f_i mod $\gtrdot$. Let $\mathcal{B} = (e_i)_{1\le i\le n}$ be the canonic basis of A^n and $\mathcal{B}' = (e'_i)_{1\le i\le n}$. Let $M = Mat_{\mathcal{B}}(\mathcal{B}')$. The reduction $\pi(M)$ of M modulo $\mathfrak{m}$ is $Mat_{\overline{\mathcal{B}}^{\mathfrak{m}}}(f_1, \dots, f_n)$. As $(f_i)_{1\le i\le n} = \pi_\sharp(B')$ is a basis of $(A/\mathfrak{m})^n$, we have $\pi(M) \in GL_n(A/\mathfrak{m})$ so $det(\pi(M)) \neq 0$ and then $det(M) \in A\backslash\mathfrak{m} = A^\times$.Thus $M \in GL_n(A)$ and $(e_i)_{1\le i\le n}$ is a basis of A^n and finally $(e_i)_{1\le i\le r}$ is a basis of P. □

Definition 1.4 (Rank of a projective module)**.** *Let P be a finitely generated projective module and $\mathfrak{p} \in Spec(A)$. Then the module $P_\mathfrak{m}$ is free and finitely generated and it's rank is called the rank of P and is noted $rk_\mathfrak{p}(P)$.*

Projective modules do not necessarily have the same rank at each prime. For example, if we take to fields K_1 and K_2 and consider the ring $A = K_1 \times K_2$ and the A module K_1, then it has rank 1 at $\{0\} \times K_1$ but rank 0 at $K_2 \times \{0\}$. We will prove later that A being a product of ring is the only type of obstruction that can happen when A is noetherian.

The following property shows that the notion of rank can be usefull :

Proposition 1.5. *Take P, Q two finitely generated projective A-modules and $\phi : P \to Q$. Then ϕ is an isomorphism if and only if ϕ is surjective and P and Q have the same rank ($\forall\mathfrak{p} \in Spec(A), rk_\mathfrak{p}(P) = rk_\mathfrak{p}(Q)$).*

Proof. The "only if" part is clear. Let prove the "if" part. As Q is projective and f surjective, we have $P \simeq Ker(f) \oplus Q$. Localizing at $\mathfrak{p}$, we get $P_\mathfrak{p} \simeq Ker(f)_\mathfrak{p} \oplus Q_\mathfrak{p}$. So $A_\mathfrak{p}^{rk_\mathfrak{p}(P)} \oplus A_\mathfrak{p}^{rk_\mathfrak{p}(Ker(f))} \simeq A_\mathfrak{p}^{rk_\mathfrak{p}(Q)}$. By hypothesis on the ranks, we thus have $rk_\mathfrak{p}(Ker(f)) = 0$. Thus $Ker(f)_\mathfrak{p} = 0$ for all $\mathfrak{p}$ so $Ker(f) = 0$ and f is injective. □

The map $rk_-(P)$ is an application from $pec(A)$ to $\mathbb{N}$, thus we have to check it's behavior with regard to Zariski's topology.

Proposition 1.6. *Let P be a finitely generated projective A-module. Then, the map $\begin{cases} Spec(A) \to \mathbb{N} \\ \mathfrak{p} \mapsto rk_\mathfrak{p}(P) \end{cases}$ is continuous.*

It's a consequence of the following lemma :

Lemma 1.7. *Take $\mathfrak{p} \in Spec(A)$, P a finitely generated projective A-module and let $n = rk_\mathfrak{p}(P)$. There exists $s \in A\backslash\mathfrak{p}$ such that $P[\frac{1}{s}] \simeq A[\frac{1}{s}]^n$. Thus, for every $\mathfrak{p}' \in Spec(A)$ not containing s, $P_{\mathfrak{p}'} \simeq A_\mathfrak{p}^n$.*

Proof. We have an isomorphism $f : A_p^n \to P_\mathfrak{p}$ and the images of elements of the canonic basis $(e_i)_{1\leq i \leq n}$ can be written $\frac{x_i}{r}$ with the $x_i \in P$ and $r \in A \backslash \mathfrak{p}$. We define $g : \begin{cases} A^n \to P \\ x \mapsto rf(x) \end{cases}$. By localizing at $\mathfrak{p}$, we get $\tilde{g} = rf$ which is an isomorphism since $r \in A_\mathfrak{p}^\times$.

Thus, the localization of $Coker(g) = P/g(A^n)$ at $\mathfrak{p}$ is trivial and since $Coker(g)$ is finitely generated there exists $t_1 \in S$ such that $t_1 Coker(g) = \{0\}$. Thus $\hat{g} : A[\frac{1}{t_1}]^n \to P[\frac{1}{t_1}]$ is surjective.

As $P[\frac{1}{t_1}]$ is projective and finitely generated, we have $P[\frac{1}{t_1}] \oplus Ker(\hat{g}) \simeq A[\frac{1}{t_1}]^n$. Localizing at $\mathfrak{p}$, we get $A_\mathfrak{p}^n \oplus Ker(\hat{g})_\mathfrak{p} \simeq P_\mathfrak{p} \oplus Ker(\hat{g})_\mathfrak{p} \simeq A_\mathfrak{p}^n$, so $Ker(\hat{g})_\mathfrak{p} = \{0\}$. As $Ker(\hat{g})$ is finitely generated, there exists $t_2 \in S$ such that $t_2 Ker(\hat{g}) = \{0\}$. Localizing a last time, we get $A[\frac{1}{t_1 t_2}]^n \to P[\frac{1}{t_1 t_2}]$. So we can take $s = t_1 t_2$.

Let $\mathfrak{p}' \in Spec(A)$ non containing s. Localizing $A[\frac{1}{s}]^n \simeq P[\frac{1}{s}]$ at $\mathfrak{p}'$, we get $A_{\mathfrak{p}'}^n \simeq P_{\mathfrak{p}'}$. □

As we have seen before, the rank of a projective module isn't necessarily the same at each prime. However, the fact that the rank is continuous and that $Spec(A)$ is connected if A is integral imply the following property.

Corollary 1.8. *Every finitely projective module on an integral domain is of constant rank.*

It would be easier to play uniquely with modules of constant rank. We will show that understanding this kind of modules is enough if A is noetherian. It is based on the following theorem

Theoreme 1.9. *Let A be a noetherian ring. Then, there exists $n \in \mathbb{N}$ and $A_1, ..., A_n$ noetherian rings with connected spectra such that*

$$A = \prod_{i=1}^{n} A_i.$$

Proof. As A is noetherian, $Spec(A)$ is the disjoint union of a finite number of irreducible components and since irreducible spaces are connected, we can write $Spec(A)$ as a disjoint finite union of clopen sets. By induction, we only need to prove that if $Spec(A) = U \sqcup V$ with U, V clopen then $A = A_1 \times A_2$ with $Spec(A_1) \simeq U$ and $Spec(A_1) \simeq V$.

We begin by proving it when A is reduced. We have $\mathcal{I}(U) + \mathcal{I}(V) = A$. Indeed, if we had $\mathcal{I}(U) + \mathcal{I}(V) \subset \mathfrak{m}$ for $\mathfrak{m}$ maximal we would have $F(\mathfrak{m}) \subset F(\mathcal{I}(U) + \mathcal{I}(V)) = F(\mathcal{I}(U)) \cap F(\mathcal{I}(V)) = U \cap V = \emptyset$. Moreover, $\mathcal{I}(U) \cap \mathcal{I}(V) = \mathcal{I}(U \cup V) = \mathcal{I}(A) = \sqrt{(0)} = \emptyset$ since A is reduced. By the chinese reminder theorem, we get $A \simeq A/\mathcal{I}(U) \times A/\mathcal{I}(V)$. Finally, $Spec(A/\mathcal{I}(U)) \simeq F(\mathcal{I}(U)) = U$ because of the correspondence between the ideals of A and A/I.

If A isn't reduced, we apply what we just did to $A/\sqrt{(0)}$. Indeed $Spec(A/\sqrt{(0)}) \simeq Spec(A) = U \sqcup V$ so we write $A/\sqrt{(0)} = B_1 \times B_2$ with $Spec(B_1) \simeq U$ and $Spec(B_2) \simeq V$. Thus there is a idempotent element $e' = (1, 0) \in B_1 \times B_2 \in A/\sqrt{(0)}$. We can then take an idempotent $e \in A$ such that $e' = e \mod \sqrt{(0)}$ (exercise) and it leads to a decomposition $A = A_1 \times A_2$ reducing to $A/\sqrt{(0)} = B_1 \times B_2 \mod \sqrt{(0)}$ so that $Spec(A_1) = Spec(B_1) = U$ and $Spec(A_2) = V$. □

We will later prove that it allows us to reduce the study of finitely generated projective modules to the modules of constant rank. But to have a good formulation of this fact, we need to introduce a new objet : the functor K_0.

2 Grothendieck's group

We first need the notion of completion of a commutative monoid.

Definition 2.1 (Completion of a commutative monoid)**.** *Let M be a commutative monoid. A completion of M is given by a pair (M^*, π) where M^* is an abelian group and $\pi : M \to M^*$ is a morphism of monoid that satisfy the following universal property :*
For any abelian group G and any morphism $f : M \to G$, there exists a unique morphism $\overline{f} : M^* \to G$ such that $f = \overline{f} \circ \pi$.

$$\begin{array}{ccc} M & \xrightarrow{f} & G \\ {\scriptstyle\pi}\downarrow & \nearrow_{\exists! \overline{f}} & \\ M^* & & \end{array}$$

commutes.

Using the universal property, a completion is unique up to isomorphism. To prove that such an object exist, we can check that $M^* := M^2/\sim$ does the work where $\sim$ is defined by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $\exists z \in M, z + x_1 + y_2 = z + x_2 + y_1$.

Looking at $M^2/\sim$, it becomes clear that $\pi(M)$ generate M^* (as a group) and that $\pi(x) = \pi(y)$ if and only if there exists $z \in M$ such that $x + y = z + y$.

We're know able to define K_0. For a ring A and $(Proj(A), \oplus)$ will be the monoid of finitely generated projective modules up to isomorphism.

Definition 2.2. *Le A be a commutative ring. It's Grothendieck's group is $K_0(A) = Proj(A)^*$.*

For example, if A is a field, a PID or a local ring, $K_0(A) \simeq \mathbb{N}^* = \mathbb{Z}$ since finitely generated modules on these rings are free and thus totally classified by their rank.

To modules $M, N \in Proj(A)$ have the same image in A if and only if there exists $P \in Proj(A)$ such that $M \oplus P = N \oplus P$. If we take $Q \in Proj(A)$ such that $P \oplus Q \simeq A^n$, it corresponds to $M \oplus A^n = N \oplus A^n$. This motivate the following definition :

Definition 2.3. *Two A-modules M, N are said to be stably isomorphic if there exists $n \in \mathbb{N}$ such that $M \oplus A^n \simeq N \oplus A^n$.*

We will see that stably isomorphic modules on Dedekind domain are isomorphic.

The following theorem that we will nether use nor prove (see [4]) allows to partially treat the notion of stably isomorphic modules.

Theoreme 2.4 (Bass)**.** *Let A be a noetherian ring of Krull dimension d and $P, M, N \in Proj(A)$. If N is of constant rank and $rk(M) > d$ and $M \oplus Q \simeq N \oplus Q$ then $M \simeq N$.*

Finally, any rank 1 stably isomorphic modules are isomorphic (we will show it later). Combined with Bass theorem, it shows that for Dedekind ring, stably isomorphic means isomorphic. We'll prove it later, independently from Bass theorem.

If $f : A \to B$ is a ring morphism we can see B as an A module thanks to f. Let M be a finitely generated projective A-module, we can then consider the finitely generated projective B-module $B \otimes_A M$. These gives a map from $Proj(A) \to Proj(B)$ that we can factorize into $f^* : K_0(A) \to K_0(B)$. This makes K_0 into a functor from the category of ring to the category of abelian groups.

For any ring A, the morphism $i : \mathbb{Z} \to A$ induces $i^* : K_0(\mathbb{Z}) \to K_0(A)$ which sends $\mathbb{Z}^n$ to A^n. If $Spec(A)$ is connected (in particular if A is an integral domain) elements of $Proj(A)$ have constant rank and we can define $f : \begin{cases} Proj(A) \to \mathbb{Z} \\ P \mapsto rk(P) \end{cases}$ extending to $f^* : K_0(A) \to \mathbb{Z}$ which is clearly surjective. As $K_0(\mathbb{Z}) \simeq \mathbb{Z}$ we have the exact sequence $0 \to Ker(f^*) \to K_0(A) \to \mathbb{Z} \to 0$ and i^* is a section. So $K_0(A) \simeq \mathbb{Z} \times Ker(\overline{f})$.

We can know complete what we have already said about modules of non constant rank and product of ring.

Proposition 2.5. *If $A = A_1 \times ... \times A_s$ is a product of rings $K_0(A) \simeq K_0(A_1) \times ... \times K_0(A_s)$ the isomorphism being given by $\pi^* : P \mapsto (\pi_1^* 1\sharp(P), ..., \pi_s^*(P))$ where π_i is the canonic projection from A to A_i.*

Proof. We just need to show that $\pi^* : \begin{cases} ProjTF_A \to ProjTF_{A_1} \times ... \times ProjTF_{A_s} \\ P \mapsto (\pi_1^*(P), ..., \pi_s^*(P)) \end{cases}$ is a morphism of monoid.

<u>Surjectivity</u>: We consider A_i-modules M_i for $i \in \{1, s\}$. Then $M = M_1 \times ... \times M_s$ is an $A_1 \times ... \times A_s$-module for the law $(a_1, ..., a_s).(m_1, ..., m_s) = (a_1 m_1, ..., a_s m_s)$. We then have $\pi_i^*(M) \simeq \pi_i^*(M_1) \times ... \times \pi_i^*(M_s)$. However as the action of A_i on M_k is trivial for $k \neq i$ we have $\pi_{i\sharp}(M_k) = 0$ for $k \neq i$. Since we also have $A_i \otimes_{A_1 \times ... \times A_n} M_i \simeq M_i$ because the action of A_i on M_i is the same than the action of $\{0\} \times ... \times A_i \times ... \times \{0\}$ on M_i. Thus $\pi_i^*(M_i) = M_i$. Then $\pi_i^*(M) = M_i$ and finally $\pi^\sharp(M) = (M_1, ..., M_s)$.

<u>Injectivity</u> : We just need to check that $P \simeq \pi_1^*(P) \times ... \times \pi_s^*(P)$ as A-modules. We consider $\phi : x \mapsto (\pi_1^*(x), ..., \pi_s^*(x))$. This morphism is surjective : indeed, we just need to check that elements of the form $(a_1 \otimes x_1, ..., a_n \otimes x_s)$ are in the image of π^* and this is since they come from $(a_1, 0, ..., 0)x_1 + ... + (0, ..., 0, a_s)x_s$. Moreover, each $\mathfrak{p} \in Spec(A)$ can be written $A_1 \times ... \times A_{i-1} \times \mathfrak{p}_i \times A_{i+1} \times ... \times A_s$ for some i and some prime $\mathfrak{p}_i$ of A_i. We have $A_\mathfrak{p} = \{0\} \times ... \times A_{\mathfrak{p}_i} \times ... \times \{0\}$ thus

$$\begin{aligned} rk_\mathfrak{p}(\pi_1^*(P) \times ... \times \pi_s^*(P)) &= rk(A_\mathfrak{p} \otimes_A \pi_1^*(P)) + ... + rk(A_\mathfrak{p} \otimes_A \pi_s^*(P)) \\ &= rk((\{0\} \times ... \times A_{\mathfrak{p}_i} \times ... \times \{0\}) \otimes_A \pi_1^*(P)) + ... + rk((\{0\} \times ... \times A_{\mathfrak{p}_i} \times ... \times \{0\}) \otimes_A \pi_s^*(P)) \\ &= rk(0) + ... + rk(0) + rk((\{0\} \times ... \times A_{\mathfrak{p}_i} \times ... \times \{0\}) \otimes_A \pi_i^*(P)) + rk(0) + ... + rk(0) \\ &= rk_f(P) \end{aligned}$$

where $f = p_i \circ \pi_i$ with p_i the canonic morphism from A_i dans $A_\mathfrak{p} = \{0\} \times ... \times A_{\mathfrak{p}_i} \times ... \times \{0\}$. As f is the inclusion $A \to A_\mathfrak{p}$ we have $rk_f(P) = rk_\mathfrak{p}(P)$. Thus, P and $\pi_{1\sharp}(P) \times ... \times \pi_{s\sharp}(P)$ have the same rank and ϕ is an isomorphism. □

For example, we have $K_0(A^n) \simeq K_0(A)^n$.

3 Picard's group

The following section is mainly taken from [2]. We won't prove or use the following theorem (see [4]), but it will help to understand why we're doing what we're doing.

Theoreme 3.1 (Serre)**.** *Le A be a noetherian ring of Krull dimension d. Let $M \in Proj(A)$ of constant rank $r > d$. Then there exists a projective module N of constant rank lower or equal to d such that $M \simeq N \oplus A^n$ for some n.*

Particularly, for Dededekind domain, it allows us to only consider modules of constant rank 1.

Lemma 3.2. *Let $P \in Proj(A)$, then $P^* = Hom_A(P, A) \in Proj(A)$ and $\forall \mathfrak{p} \in Spec(A), rk_{\mathfrak{p}}(P) = rk_{\mathfrak{p}}(P^*)$.*

Proof. If $P \oplus Q \simeq A^n$ then $A^n \simeq Hom_A(A, A)^n \simeq Hom_A(A^n, A) \simeq Hom_A(P \oplus Q, A) \simeq Hom_A(P, A) \oplus Hom_A(Q, A)$ so P^* is finitely generated and projective.

If $\mathfrak{p} \in Spec(A)$, we have $(P^*)_{\mathfrak{p}} = (Hom_A(P, A))_{\mathfrak{p}} \simeq Hom_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, A_{\mathfrak{p}}) \simeq Hom_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}^{rk_{\mathfrak{p}}(P)}, A_{\mathfrak{p}}) \simeq Hom_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}, A_{\mathfrak{p}})^{rk_{\mathfrak{p}}(P)} \simeq A_{\mathfrak{p}}^{rk_{\mathfrak{p}}(P)}$ the first localization not being as trivial as it seems (it's only true because projective modules are finitely presented). Therefore $rk_{\mathfrak{p}}(P) = rk_{\mathfrak{p}}(P^*)$ □

It allows us to define the Picard's group of a ring.

Definition 3.3 (Picard's Group)**.** *The set of finitely generated projective A-modules of rank 1 (up to isomorphism) is a group for the tensor product. It's called the Picard's group and we write $Pic(A)$.*

Proof. Neutral element and product stability are easy to check. For inverse, we need to show that $L \otimes L^* = A$ (L being in $Pic(A)$ by the precedent lemma). We consider the morphism characterized by $\phi : f \otimes m \mapsto f(m)$. For all $\mathfrak{p} \in Spec(A)$, we have $(L^* \otimes_A L)_{\mathfrak{p}} \simeq (L_{\mathfrak{p}}^* \otimes_{A_{\mathfrak{p}}} L_{\mathfrak{p}})$ allow us to see $\phi_{\mathfrak{p}}$ as $\begin{cases} L_{\mathfrak{p}}^* \otimes_{A_{\mathfrak{p}}} L_{\mathfrak{p}} \to A_{\mathfrak{p}} \\ f \otimes m \mapsto f(m) \end{cases}$. Since $L_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$, $\phi_{\mathfrak{p}}$ is an isomorphism. Thus ϕ is an isomorphism. □

Let A be an integral domain. A fractional ideal I of A is a sub-A-module of K such that there exists $a \in K^\times$ such that aI is an ideal of A. We extend the definition of the product of ideals to fractional ideals giving a structure of monoid to the set of fractional ideal.

Definition 3.4 (Cartier's Group)**.** *The group of invertible fractional ideals of A is called Cartier's Group of A and we write $Cart(A)$.*

The subset of principal fractional ideals (i.e. of the form aA with $a \in K^\times$) is noted $\mathcal{P}(A)$.

Proposition 3.5. *We have $Cart(A)/\mathcal{P}(A) \simeq Pic(A)$.*

Proof. Let $I \in Cart(A)$. We have allready shown that it is projective. Since A is an integral domain, I is of constant rank and localizing at (0), we have $rk(I) = dim_K(I \otimes_A K) = dim_K(K) = 1$.

Thus, we have a natural map from $Cart(A)$ to $Pic(A)$. We need to check it is a morphism : it should satisfy $\forall I, J \in Cart(A), IJ \simeq I \otimes_A J$. The map $\begin{cases} I \times J \to IJ \\ (x, y) \mapsto xy \end{cases}$ is bilinear so we get a linear map $I \otimes_A J \to IJ$which is surjective by definition of IJ. Since $I \otimes_A J$ and IJ are projective of rank 1, it's an isomorphism.

For the injectivity, if $I \in Cart(A)$ is such that $I \simeq A$ as an A-module, we can take an isomorphism $\phi : A \to I$ and we have $I = \phi(1)A$ so $I \in \mathcal{P}(\mathcal{A})$.

Finally, for the surjectivity, we need to show that every $L \in Pic(A)$ is isomorphic to an element of $Cart(A)$. As $rk(L) = 1$,we have $L \otimes_A K \simeq K$. As a projective module is flat, the injection $A \subset K$ gives an injection from $L \otimes_A A$ to $L \otimes_A K$ and thus an injection from L to K so we get a fractional ideal of A isomorphic to L that we note I. Doing the same thing with L^*, there is a fractional ideal J such that $L^* \simeq J$. We then have $IJ \simeq I \otimes_A J \simeq L \otimes_A L^* \simeq A$ so $I \in Cart(A)$. □

You can check that the proof also shows that any projective fractional ideal of A is invertible.

4 Finitely generated modules on a Dedekind domain

In this section, we will classify finitely generated modules on Dedekind domain. We will proceed in free steps :

- Prove that stably isomorphic means isomorphic for Dedekind domains.
- Prove that $K_0(A) \simeq \mathbb{Z} \oplus Cl(A)$ where $Cl(A) = Cart(A)/\mathcal{P}(A)$.
- Show that any modules decompose into the sum of a projective module and a torsion module.

- Show that torsion modules are direct sum of quotient of A.

The following propositions come from [1].

Proposition 4.1. *Let A be a Dedekind domain and P a finitely generated projective A-module. There exists ideals $I_1, \ldots, I_n$ of A such that $P \simeq I_1 \oplus \ldots \oplus I_n$.*

Proof. Let $P \oplus Q = A^n$. Projection on the last coordinate gives a map $\pi : P \to A$ with $ker(\phi) \subset A^{n-1}$. $\phi(P) = I_n$ is an ideal of A which is Dedekind, so it is invertible hence projective. Thus $P \simeq Ker(\phi) \oplus I_k$ with $Ker(\phi) \subset A^{n-1}$ being projective. We complete the proof by induction. □

Proposition 4.2. *Let A be an integral domain. If two directs sums of non zeros ideals $I_1 \oplus \cdots \oplus I_r$ and $J_1 \oplus \ldots \oplus J_s$ are isomorphic then $s = r$ and $I_1 \cdots I_r$ and J_{1s} are in the same ideal class. If A is Dedekind, the converse is true.*

Adjoined with the precedent proposition, it shows in particular that stably isomorphic A-modules are isomorphic.

Proof. If $\phi : I \to J$ is an morphism, there exists a unique $a \in K$ such that $\forall x \in I, \phi(x) = ax$. Indeed, for all $a, x \in I$, $x\phi(a) = \phi(ax) = a\phi(x)$ so $\forall x \in I, \phi(x) = \frac{\phi(a)}{a}x$. Thus, an isomorphism $I_1 \oplus \cdots \oplus I_r \simeq J_1 \oplus \ldots \oplus J_s$ is characterized by an $r \times s$ matrice $M = (a_{ij})$ such that $\phi(a_1, \ldots, a_r) = (b_1, \ldots, b_s)$ where $b_i = \sum q_{ij} a_j$. If as ϕ is an isomorphism, M is invertible so $r = s$. We will know prove that $det(Q) I_1 \cdots I_r = J_1 \cdots J_s$. If we have $a_i \in I_i$, $det(Q) a_1 \cdots a_r = det(M)$ where $M = Q \times diag(a_1, \cdots, a_r)$ whose coefficients of the i^{th} row are in J_i so $det(Q) I_1 \cdots I_r \subset J_1 \cdots J_s$. We then prove the converse reasoning on Q^{-1}.

To prove the converse when A is a Dedekind domain, we clearly just need the following lemma. □

Lemma 4.3. *if A is a Dedekind domain then $I_1 \oplus I_2 \simeq A \oplus I_1 I_2$ where $I_1 I_2$ are integral ideals.*

Proof. The kernel of the surjective map $I_1 \oplus I_2 \to A$ such that $(a_1, a_2) \to a_1 + a_2$ is isomorphic to $I_1 \cap I_2$. Thus $I_1 \oplus I_2 \simeq A \oplus I_1 \cap_2$. If I_1 and I_2 are coprime, this concludes. Else, we just need to replace I_1 by an integral ideal in the same class than I_1 and prime to I_2. Choose $a \in I_1$ and write $aA = I_1 J$. J/JI is an ideal of A/JI_2 so is of the form xA/JI_2 and $J = JI_2 + xA$. Multiplying by I_1 and dividing by a, we have $A = I_2 + \frac{x}{a} I_1$ proving that such an ideal exists. □

Proposition 4.4. *If A is a Dedekind domain, then $K_0(A) \simeq (Z) \times Cl(A)$.*

Proof. The two last proposition show that we have a natural morphism of monoid $Proj(A) \to \mathbb{Z} \oplus Cl(A)$ which maps M to $(rk(M), I_1 \cdots I_n)$ where $M = I_1 \oplus \cdots \oplus I_n$. We can factor it into a morphism $K_0(A) \to \mathbb{Z} \oplus Cl(A)$. The fact that it is an isomorphism is pretty straightforward. □

We know treat the more general case of finitely generated modules on Dedekind rings. We will need the following proposition that we won't prove here (see [4]).

Proposition 4.5. *Let A be any commutative ring. An A-module is projective and finitely generated if and only if it is locally free and finitely presented.*

The following proposition are taken from [3].

Proposition 4.6. *Let A be a Dedekind domain. Then a finitely generated module on A is projective if and only if it is torsion free.*

Proof. A finitely generated module on a noetherian ring ring is finitely presented so we just need to check that for all$\mathfrak{p} \in Spec(A)$, $M_\mathfrak{p}$ is free. As A is Dedekind, $A_\mathfrak{p}$, thus $M_\mathfrak{p}$ is free if and only if it is torsion free. Since $(M_\mathfrak{p})^{tors} = (M^{tors})_\mathfrak{p}$, it it the case. □

Proposition 4.7. *Let M be a finitely generated module on a Dedekind domain. Then $M \simeq M^{tors} \oplus P$ where P is finitly generated and projective and M^{tors} is the torsion of M.*

Proof. We have the exact sequence $0 \to M^{tors} \to M \to M/M^{tors} \to 0$ which splits since M/M_{tors} has no torsion so is projective. □

Proposition 4.8. *Le A be a Dedekind domain and M a finitely generated A-module then $M^{tors} \simeq \bigoplus_{\mathfrak{p} \in SpM(A)} (M_\mathfrak{p})^{tors}$.*

Proof. As M is finitely generated and A noetherien, $M^{tors} \subset M$ is finitely generated. If $m_1, ..., m_n$ generates $M^t ors$ and $a_1, ..., a_n \in \mathbb{N}$ are non zero and such that $\forall i \in [|1, n|], a_i m_i = 0$. Then for $I = (a_1 ... a_n)$, we have $IM^{tors} = \{0\}$. We can write $\mathfrak{p} = \prod_{i=1}^{n} \mathfrak{p}_i^{\alpha_i}$ so that for all $\mathfrak{p} \in SpM(A)$ not being a $\mathfrak{p}_i$, I and $\mathfrak{p}$ are prime. Take $t \in I \backslash \mathfrak{p}$, we have $tM^{tors} = \{0\}$ thus $(M^{tors})_\mathfrak{p} = \{0\}$.
So we can define a morphism from M^{tors} to $\bigoplus_{\mathfrak{p} \in SpM(A)} (M_\mathfrak{p})^{tors}$. As it is an isomorphism after localizing at every $\mathfrak{p} \in SpM(A)$, it's an isomorphism. □

From the classification of modules on PID, we deduce :

Corollary 4.9. *Let A be a Dedekind domain and M a finitely generated A-module, there exists unique ideals $I_1 \subset ... \subset I_n$ of A such that*

$$M_{tors} \simeq A/I_1 \oplus ... \oplus A/I_n$$

.

And putting everything together :

Theoreme 4.10 (Finitely generated modules on a Dedekind domain)**.** *Let A a Dedekind ring and M a finitely generated A-module. Then, there exists a unique $r \in \mathbb{N}$, a unique ideal class $[I]$ (such that $[I] = [A]$ if $r = 0$) and unique proper ideals $I_1 \subset ... \subset I_n$ of A such that :*

$$M \simeq A^{r-1} \oplus I \oplus A/I_1 \oplus ... \oplus A/I_n$$

.

With the convention $A^{-1} \oplus A = \{0\}$.

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