Lecture group on algebraic K-theory
Session 2: K_1 and K_2 of a ring

15/10/25

Throughout this document, R denotes an associative and commutative ring with unit, but a lot of what is presented here can be generalized to non-commutative rings. The goal of this session is to say a bit about K_1 and K_2, so the next sessions can move into more homotopical territory.
We mostly follow Srinivas, occasionally borrowing things from Milnor.

Addendum on K_0 : the Eilenberg swindle. When defining $K_0(R)$ of a ring, we restricted ourselves to only looking at projective modules that are *finitely generated*. This is important for set-theoretic concerns (as isomorphism classes of all projective modules over R do not form a set), but there is also a purely algebraic reason for this. Indeed, let P be a finitely generated projective module over R. We know that we can find another R-module Q such that $P \oplus Q \simeq R^n$ for some $n \in \mathbb{N}$. Let R^∞ be the R-module $R^(\mathbb{N})$ (the union of all $R^k, k \in \mathbb{N}$). Then,

$$R^n \oplus R^n \oplus \cdots \simeq R^\infty$$

So, we obtain

$$(P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \simeq \mathbb{R}^\infty$$

We then get the following isomorphisms:

$$\begin{aligned} R^\infty &\simeq (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\ &\simeq P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\ &\simeq P \oplus R^\infty \end{aligned}$$

If we allowed infinite-type projective modules in $K_0(R)$, we would get $[R^\infty] = [P] + [R^\infty]$, hence $[P] = 0$ for all finitely generated projective modules P, which is clearly not something we desire.

Remark. This trick is named an *Eilenberg swindle*, after Samuel Eilenberg.

Definition of $K_1(R)$. For $n \in \mathbb{N}^*$, consider the group $\mathrm{GL}_n(R)$. We have an embedding $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, so we can consider the colimit

$$\begin{aligned} \mathrm{GL}(R) &= \mathrm{colim}_{n \in \mathbb{N}^*} \mathrm{GL}_n(R) \\ &= \mathrm{colim}(\mathrm{GL}_1(R) \hookrightarrow \cdots \hookrightarrow \mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R) \hookrightarrow \cdots) \end{aligned}$$

This is again a group.

Definition 1. We define $K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$. Note that is functorial with respect to ring maps.

Let $E_n(R)$ be the group of elementary matrices, that is the group generated by matrices of the form

$$e_{ij}^{(n)}(\lambda) = I_n + \lambda E_{i,j}, \quad 1 \leq i \neq j \leq n \text{ and } \lambda \in R$$

This is clearly a subgroup of $\mathrm{GL}_n(R)$. The embedding $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ gives an embedding $E_n(R) \hookrightarrow E_{n+1}(R)$, and again we can consider the colimit

$$E(R) = \mathrm{colim}_{n \in \mathbb{N}^*} E_n(R)$$

Under the embedding $E_n(R) \hookrightarrow E_{n+1}(R)$, $e_{ij}^{(n)}$ becomes $e_{ij}^{(n+1)}(\lambda)$. We can therefore define $e_{ij}(\lambda) \in E(R)$ as the image of all $e_{ij}^{(n)}(\lambda)$ for all $n \geq i, j$.

Lemma 2. *For all $\lambda, \mu \in R$, we have the following identities:*

1. $e_{ij}(\lambda)e_{ij}(\mu) = e_{ij}(\lambda + \mu)$.
2. $[e_{ij}(\lambda), e_{kl}(\mu)] = 1$ *when* $j \neq k, i \neq l$.
3. $[e_{ij}(\lambda), e_{jk}(\mu)] = e_{ik}(\lambda\mu)$ *when* $i \neq k$.

In particular, 3. implies that $E(R) = [E(R), E(R)]$.

This lemma results from straightforward computations.

Proposition 3. *We have $E(R) = [GL(R), GL(R)]$.*

Proof. Let $A, B \in \mathrm{GL}_n(R)$. Then, in $\mathrm{GL}_{2n}(R)$, we have

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

Notice that if $C \in M_n(R)$, the matrices $\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix}$ and $\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}$ are in $E_{2n}(R)$ (use the $e_{ij}^{(2n)}$ to perform repeated transvections). Then, the formula

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ A^{-1} - I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ A - I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & -A^{-1} \\ 0 & I_n \end{pmatrix}$$

allows us to show $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] \subset E_{2n}(R)$, which concludes the proof. $\square$

Interpretation of the definition. Our definition of $K_0(R)$ makes it clear that it encodes projective modules up to isomorphism and stability. If we're thinking homotopically, we think of $K_0(R)$ as the π_0 of something, and $K_1(R)$ should encode loops. Thinking of isomorphisms as paths makes sense in this interpretation, as $K_0(R)$ identifies modules connected by isomorphisms. Loops of isomorphisms should clearly be automorphisms. To make that into a group, we need these automorphisms to live in the same place, but fortunately, all projective modules are summands of $R^\infty = R^{\{}(\mathbb{N})\}$, which motivates considering $\mathrm{GL}(R)$. Therefore, we should have an morphism $p : \mathrm{GL}(R) \to K_1(R)$. Now, an element of $\mathrm{GL}(R)$ can be represented by a matrix A living in some $\mathrm{GL}_n(R)$. Because we want $K_1(R)$ to also be stable, we have

$$p(A \oplus I) = p(A) = p(I \oplus A)$$

which implies

$$p(A)p(B) = p(A \oplus I)p(I \oplus B) = p((A \oplus I)(I \oplus B)) = p(A \oplus B)$$

but also

$$p(A)p(B) = p(I \oplus A)p(B \oplus I) = p(B \oplus A) = p(B)p(A)$$

hence, p must factor through $\mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$, which motivates our definition of K_1.
I found this interpretation in an expository article by Zakharevich.

Lemma 4. *Let P be a finitely generated projective module over R. There is a natural map $\mathrm{Aut}(P) \to K_1(R)$.*

Proof. First, choose another finitely generated projective module Q such that $P \oplus Q$ is free, and pick a basis $b_1, \ldots, b_n$ for $P \oplus Q$. Using this basis, we get a map

$$\begin{array}{ccccc} \mathrm{Aut}(P) & \longrightarrow & \mathrm{Aut}(P \oplus Q) & \longrightarrow & \mathrm{GL}_n(R) \\ f & \longmapsto & f \oplus \mathrm{id}_Q & \longmapsto & \mathrm{mat}_{(b_1,\ldots,b_n)}(f \oplus \mathrm{id}_Q) \end{array}$$

that we can then compose with $\mathrm{GL}_n(R) \to K_1(R)$ to get a map $\mathrm{Aut}(P) \to K_1(R)$. We need to check it is well-defined. Assume we have another basis $b'_1, \ldots, b'_m$ for $P \oplus Q$(eventually $m \neq n$). Let $C = (c_{ij}) \in M_{m,n}(R)$ be the matrix defined by

$$b'_i = \sum_{j=1}^{n} c_{ij} b_j$$

which is necessarily invertible. Let A be the matrix of $f \oplus \mathrm{id}_Q$ with respect to $(b_1, \ldots, b_n)$ for some $f \in \mathrm{Aut}(P)$. In the basis $(b'_1, \ldots, b'_m)$, $f \oplus \mathrm{id}_Q$ is represented by the matrix $CAC^{-1} \in \mathrm{GL}_m(R)$. Then, in $\mathrm{GL}_{n+m}(R)$, we have

$$\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} CAC^{-1} & 0 \\ 0 & I_n \end{pmatrix}$$

Hence, changing the basis doesn't affect the image of an automorphism of P in $K_1(R)$. Now, if we pick another module Q' such that $P \oplus Q' \simeq R^m$, then $Q \oplus R^m \simeq Q' \oplus R^n$, so letting $Q'' = Q \oplus R^m$, we have arrows

$$\mathrm{Aut}(P) \to \mathrm{Aut}(P \oplus Q) \to \mathrm{Aut}(P \oplus Q'') \xrightarrow{\sim} \mathrm{GL}_{n+m}(R)$$

and

$$\mathrm{Aut}(P) \to \mathrm{Aut}(P \oplus Q') \to \mathrm{Aut}(P \oplus Q'') \xrightarrow{\sim} \mathrm{GL}_{n+m}(R)$$

which are the same up to an automorphism of R^{n+m}, hence the same in $K_1(R)$. This completes the proof. $\square$

Remark. If R is a field, this definition gives back the determinant.

K_1 of a local ring. We can compute the K_1 of a local ring:

Proposition 5. *I R is a commutative local ring, then $K_1(R) \simeq R^*$.*

Proof. Consider the natural map $R^* \to \mathrm{GL}_1(R) \to K_1(R)$. We first show it is surjective. Let $A = (a_{ij}) \in \mathrm{GL}_n(R)$ for some $n \in \mathbb{N}^*$. Let $\mathfrak{m}$ be the unique maximal ideal of R. That A is invertible in $M_n(R)$ shows it is invertible in $M_n(R/\mathfrak{m})$. Since $R/\mathfrak{m}$ is a field, each line and column of A must be nonzero in $R/\mathfrak{m}$, i.e. have an invertible coefficient (because R is local). Therefore, we can perform a column operation and assume $a_1 1 \in R^*$, because column operations do not change the image of A in $K_1(R)$, since $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$. Now that we have $a_{11} \in R^*$, we can do column operations to ensure $a_{1i} = 0$ for $i > 1$. On the second row, we can similarly assume $a_{22} \in R^*$, then make $a_{21} = 0$ using that $a_{11} \in R^*$ and a row operation, then use $a_{22} \in R^*$ to get $a_{2i} = 0$ for $i > 2$. Continuing this process, we have show, that we can assume A is diagonal without changing its image in $K_1(R)$. It remains to show that diagonal matrices are in the image of $R^* \to \mathrm{GL}_1(R) \to K_1(R)$. We will use the fact (Whitehead's lemma) that $\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \in E_{2n}(R)$ for any $B \in \mathrm{GL}_n(R)$. Eventually applying an embedding into some $\mathrm{GL}_{2^k}(R)$, we can assume A is of size a power of 2. Then, it is not hard to see that we successively multiply by matrices in $E_{2^k}(R)$ to get a diagonal matrix whose diagonal entries are all 1 except the first one, which shows A is in the image of $R^* \to K_1(R)$, which proves surjectivity.
To show injectivity of this map, it suffices to notice that determinants $\mathrm{GL}_n(R) \to R^*$ glue to a map $\mathrm{GL}(R) \to R^*$ which factorises to a determinant map $K_1(R) \to R^*$, and that the composition $R^* \to K_1(R) \xrightarrow{\det} R^*$ is id_{R^*}. □

Remark. In his book, Srinivas proves a more general result, which is that if R is a local ring that is not necessarily commutative, we have $K_1(R) = (R^*)^{\mathrm{ab}}$. This requires more technical details.

Exact sequences. Historically, K_0 was defined by Grothendieck, K_1 by Bass, and K_2 by Milnor. The group K_1 was named like so because it fits into exact sequences that make K_* look like a homology theory, and that motivates trying to find higher K-groups. Concretely, we have the following.

Proposition 6 ("Mayer-Vietoris" sequence)**.** *Consider a pullback of rings*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"] & R' \end{tikzcd}$$

and assume either j_1 or j_2 is surjective. Then, there is an exact sequence

$$K_1(R) \to K_1(R_1) \oplus K_1(R_2) \to K_1(R') \to K_0(R) \to K_0(R_1) \oplus K_0(R_2) \to K_0(R')$$

Remark. Like in the topological equivalent, the morphisms above are given by

$$K_i(R) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_i(R_1) \oplus K_i(R_2) \xrightarrow{(j_1)_* \oplus -(j_2)_*} K_i(R')$$

The connecting homomorphism is harder to define (it requires knowing how to glue projective modules over R_1 and R_2 together).

Remark (Another glimpse of an exact sequence). If R is a Dedekind ring with fraction field F, the computation $K_0(R) = \mathbb{Z} \oplus \mathrm{Cl}(R)$ gives the following exact sequence:

$$R^* \to F^* \to \bigoplus_{\mathfrak{p} \text{ prime}} \mathbb{Z} \to K_0(R) \to K_0(F) \to 0$$

We know that $K_0(R/\mathfrak{p}) = \mathbb{Z}$ because $R/\mathfrak{p}$ is a field since R is Dedekind. We also know that $K_1(F) = F^*$ since F is a field hence local. Since the determinant gives a surjective map $K_1(R) \to R^*$, we get an exact sequence

$$K_1(R) \to K_1(F) \to \bigoplus_{\mathfrak{p} \text{ prime}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(F) \to 0$$

Decomposition with SK_1. The determinant $K_1(R) \xrightarrow{\det} R^*$ gives a short exact sequence

$$0 \to SK_1(R) \to K_1(R) \xrightarrow{\det} R^* \to 0$$

Where $SK_1(R) = \mathrm{SL}(R)/[\mathrm{SL}(R), \mathrm{SL}(R)]$. It is not hard to see that $[\mathrm{SL}(R), \mathrm{SL}(R)] = E(R)$, so in fact $SK_1(R) = \mathrm{SL}(R)/E(R)$.

Example 7. We give no proof for those results.

1. If R is Noetherian of Krull dimension ≤ 1 with finite residue fields at all maximal ideals, then $SK_1(R)$ is torsion.

2. If R is a Euclidean domain, then $SK_1(R) = 0$.

3. If R is the ring of integers of a number field (finite extension of $\mathbb{Q}$), then $SK_1(R) = 0$.

Definition of K_2. The n-th *Steinberg group* of R is defined using the following presentation:

$$St_n(R) = \left\langle x_{ij}^{(n)}(\lambda) \quad \begin{matrix} 1 \leq i, j \leq n \\ i \neq j \\ \lambda \in R \end{matrix} \;\middle|\; \begin{matrix} x_{ij}^{(n)}(\lambda) x_{ij}^{(n)}(\mu) = x_{ij}^{(n)}(\lambda + \mu) & \\ [x_{ij}^{(n)}(\lambda), x_{kl}^{(n)}(\mu)] = 1 & i \neq \ell, k \neq j \\ [x_{ij}^{(n)}(\lambda), x_{jk}^{(n)}(\mu)] = x_{ik}^{(n)}(\lambda\mu) & i \neq k \end{matrix} \right\rangle$$

Lemma 2 tells us we have a natural surjection $St_n(R) \to E_n(R)$, where $x_{ij}^{(n)}(\lambda)$ is sent to $e_{ij}^{(n)}(\lambda)$. We also have natural homomorphisms $St_n(R) \to St_{n+1}(R)$ (send $x_{ij}^{(n)}(\lambda)$ to $x_{ij}^{(n+1)}(\lambda)$). We can therefore consider the colimit (the infinite Steinberg group):

$$St(R) = \mathrm{colim}_{n \in \mathbb{N}^*} St_n(R)$$

and there is a surjection $St(R) \to E(R)$.

Definition 8. We define $K_2(R)$ to be the kernel of the surjection $St(R) \to E(R)$.

The following proposition requires knowing about central extensions and group homology. I'm including it here as motivation for K_2.

Proposition 9. *We have* $K_2(R) = H_2(E(R), \mathbb{Z}) = \operatorname{colim}_n H_2(E_n(R), \mathbb{Z})$. *Moreover, the "universal central extension of* $E(R)$*" is*

$$0 \to K_2(R) \to St(R) \to E(R) \to 0$$

in the sense that it is initial among central extensions of $E(R)$.

Module structures over K_0. We can make K_0, K_1 and K_2 into modules over K_0.

Proposition 10. *The abelian group* $K_0(R)$ *can be endowed with a product operation that makes it into a commutative and associative ring with identity element* $[R]$. *Moreover, the group of invertible elements is precisely* $\operatorname{Pic}(R)$, *the Picard group of* R.

Proof. A tensor product of projective modules is again projective, as

$$\operatorname{Hom}(P \otimes_R Q, -) \simeq \operatorname{Hom}(P, \operatorname{Hom}(Q, -))$$

and composing two exact functors gives an exact functor. Since a tensor product of finitely generated modules is again finitely generated and the isomorphism class of $P \otimes Q$ depends only on those of P and Q, we get a well-defined product

$$K_0(R) \otimes_{\mathbb{Z}} K_0(R) \to K_0(R)$$

Associativity, commutativity and the fact that $[R]$ is an identity come from well-known properties of the tensor product. That $(K_0(R))^\times = \operatorname{Pic}(R)$ is easy to see once one realizes being invertible forces a module to be of rank 1 everywhere. $\square$

Proposition 11. *The abelian group* $K_1(R)$ *is naturally a module over* $K_0(R)$.

Proof. Pick a projective module P. For $n \in \mathbb{N}$, consider the map

$$\begin{array}{rcl} \mathrm{GL}_n(R) & \to & \operatorname{Aut}(P \otimes R^n) \\ f & \mapsto & \mathrm{id}_P \otimes f \end{array}$$

Since P is projective, $P \otimes R^n$ is also projective, and lemma 4 gives us a map $\operatorname{Aut}(P \otimes R^n) \to K_1(R)$, hence we get a composite map $h_P^n : \mathrm{GL}_n(R) \to \operatorname{Aut}(P \otimes R^n) \to K_1(R)$. If P' is also a projective module, we have $h^n_{P \oplus P'} = h^n_P + h^n_{P'}$, which shows h^n_P only depends on the class of P in $K_0(R)$. Passing to the direct limit, we get $h_P : K_1(R) \to K_1(R)$, and the desired map is $h_\bullet : K_0(R) \otimes_{\mathbb{Z}} K_1(R) \to K_1(R)$. $\square$

The abelian group $K_2(R)$ is naturally a module over $K_0(R)$. The proof is in Srinivas.

A glimpse at the fundamental theorem of algebraic K-theory. The fundamental theorem of algebraic K-theory relates the K-theory of R, of $R[t]$ and of $R[t,t^{-1}]$. For a regular[1] ring R, we have that $K_i(R[t]) = K_i(R)$ and $K_i(R[t,t^{-1}]) = K_i(R) \oplus K_{i-1}(R), i \geq 0$ with $K_{-1}(R) = 0$. Bass proved it independently for K_1 first and Quillen extended it later. For now, we can say the following. Localization gives a morphism $\alpha : K_0(R) \to K_0(R[t,t^{-1}])$. Multiplication by t is an automorphism of $R[t,t^{-1}]$, hence gives a class in $K_1(R[t,t^{-1}])$. Therefore, we get a morphism $\psi : \begin{array}{ccc} K_0(R) & \to & K_1(R[t,t^{-1}]) \\ [P] & \mapsto & \alpha([P]) \cdot [t] \end{array}$. It turns out this morphism is always injective and that its image is functorially a direct summand in $K_1(R[t,t^{-1}])$.

Matsumoto's theorem and an application. The following theorem gives a presentation of $K_2(F)$, where F is a field.

Theorem 12 (Matsumoto)**.** *If F is a field, then*

$$K_2(F) \simeq \mathbb{Z}\left[\{a,b\}, a,b \in F^* \middle| \begin{array}{ll} \{a_1a_2,b\} = \{a_1,b\}\{a_2,b\} & \forall a_1,a_2,b \in F^* \\ \{a,b\} = \{b,a\}^{-1} & \forall a,b \in F^* \\ \{a,1-a\} = 1 & \forall a \in F^* \setminus \{1\} \end{array}\right]$$

Remark. We use multiplicative notation, but the group is abelian.

Remark. More generally, a *symbol* on F with values in an abelian group G is a map $(-,-) : F^* \times F^* \to G$ satisfying axioms looking like the presentation of $K_2(F)$ above. In a way, $K_2(F)$ is the free abelian group on symbols on F. Various symbols of interest are showcased in Srinivas (examples 1.15 - 1.18).

Proposition 13. *If F is a finite field, then $K_2(F) = 0$.*

Proof. Recall that F^* is cyclic because F is a finite field. Let $u \in F^*$ be a generator. Because $\{a_1a_2,b\} = \{a_1,b\}\{a_2,b\}$ and $\{a,b\} = \{b,a\}^{-1}$, we only need to prove $\{u,u\} = 0$. If F has characteristic 2, then $\{u,u\} = \{u,-u\}$. In general, we have that

$$\{x,y\}^{-1} = \{x,y^{-1}\}$$

and

$$\{x,-x\} = \{x,1-x\}\{x,1-x^{-1}\}^{-1} = \{x^{-1},1-x^{-1}\}^{-1} = 1$$

hence if F has characteristic 2, we are done. We now assume F has odd characteristic. Let $q = |F|$. Then, $u^{(q-1)/2} = -1$, so

$$\{u,u\}^{(q+1)/2} = \{u,u^{(q+1)/2}\} = \{u,-u\} = 1$$

Hence, $\{u,u\}$ must have an order diving $(q+1)/2$, but also $q-1$ (since $u^{q-1} = 1$). We conclude that $\{u,u\}$ has order at most 2. Now, consider the bijection $v \mapsto 1-v$, of $F^* \setminus \{1\}$ onto itself. There are $(q-1)/2$ squares and non-squares in F^*. Since 1 is a square, the number of non-squares in $F^* \setminus \{1\}$ is $(q-1)/2$, and the number of squares is $(q-3)/2$.

[1] A ring is *regular* if it is Noetherian, and if its localization at any prime ideal has the property that the minimal number of generators of its maximal ideal is equal to its Krull dimensions.

We deduce that there must exist a non-square $v \in F^* \setminus \{1\}$ such that $1 - v$ is also not a square. Then, $v = u^i, 1 - v = u^j$ with i, j odd. Then,

$$\{u, u\}^{ij} = \{u^i, u^j\} = \{v, 1 - v\} = 1$$

which shows $\{u, u\}$ has odd order, hence $\{u, u\} = 1$. This concludes. $\square$