Higher Algebraic K-Theory groups (part I): Quillen's Plus construction

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Abstract

The main goal of this document is introducing the notion of **Quillen's Plus Construction** and showing that it may be a solid generalization to the classical K-Theory developed by hand (in other words: this new sequence of groups will coincide with the already seen K_1, K_2 groups). For the sake of recollection, these are the notes regarding the fourth meeting of the GT of Algebraic K-Theory organized for the "Master 2 en Mathematiques Fondamentales" of IMJ-PRG and its main content directly comes from the book "Algebraic K-Theory" by V. Srinivas[1].

Classifying Space of a small category and useful functors

Let us briefly recall a very standard notion of algebraic topology and category theory: the classifying spaces of a small category.

Def. *The classifying space of a category $\mathcal{C}$ is the geometric realization of its nerve.*

To make it make sense, the notion of nerve of a category and of geometric realization need to be introduced.
We consider the category Δ of isomorphism classes of finite linear orders, whose objects are the ordered sets $n := \{0 < 1 < 2 < \cdots < n\}$ and whose morphisms $n \to m$ are increasing maps (not necessarily strictly).
Of those maps, certains are simpler than others: consider the $(n+1)$ *injective* maps $n-1 \to n$ which skips the index $j \in \{0, \dots, n\}$, we will call them *face maps* and denote them $(\partial_j^n)_{j \in n}$.
Similarly, there are n surjective maps $n \to n-1$ repeating the index j which we will call *degeneracy maps* and denote by $(s_j^{n-1})_{j \in n}$.
It is clear that injective maps between the objects are compositions of the face maps and surjective maps are compositions of the degeneracy maps: by the epi-mono factorization in **Set**, one gets that the morphisms of Δ are compositions of these important maps.
We now consider a small category $\mathcal{C}$ and call **a simplicial object in** $\mathcal{C}$ a functor $\Delta^{op} \xrightarrow{F} \mathcal{C}$.
The image $F(n)$ should be thought of as n-skeleton of the data F and the image of the face maps tells us how higher skeleta glue on lower ones (viceversa for the degeneracy map). Note that - by contravariancy - one ought call the images $F(\partial_\bullet^\bullet), F(s_\bullet^\bullet)$ respectively co-face and co-degeneracy.
We will call the category of simplicial objects on **Set** simply **sSet** as simplicial sets.
Let us now go through a fundamental example.
Given X a topological space, consider the data $S_\bullet(X) = (S_n(X))_{n \in \mathbb{Z}}$ where $S_n(X)$ is the <u>set</u> of

(continuous) maps $\Delta_n \to X$ where the first space is the standard simplex of $\mathbb{R}^{n+1}$ (namely, the subspace of nonnegative coordinates summing to 1).
We can make the assignment $n \mapsto S_n(X)$ into a simplicial set by defining suitable maps $S_m(X) \overset{m \mapsto n}{\longleftarrow} S_n(X)$.
This is easily done by finding functorial maps $\Delta_m \to \Delta_n$ and noting that $S_n(X) = Hom_{\mathrm{Top}}(\Delta_n, X)$ is contravariant. To achieve the first goal, given $f : m \to n$ one could create $\tilde{f} : \Delta_m \to \Delta_n$ by the formula

$$(\tilde{f}((x_0, \dots, x_m)))_i = \sum_{j \in f^{-1}(i)} x_j$$

for each $0 \le i \le n$, with the clear "convention" that sums over an empty index set is zero.
One checks this does indeed define a functor, the so called "set of singular chains functor". Taking its linearization with respect to a $k-$algebra[1] R one obtains the more used "module of singular chains with coefficient in R".
Now we will introduce the second functor of this section, the geometric realization functor.
Given a simplicial set $F : \Delta^{op} \to \mathbf{Set}$, one can construct a topological space $|F|$ as follows:

$$\left(\bigsqcup_{n \in \mathbb{Z}} F(n) \times \Delta_n \right) / \sim$$

where $\sim$ is generated by $(x, \tilde{f}(y)) \sim (F(f)x, y)$ for each $f : m \to n$, for each $x \in F(n)$ and for each $y \in \Delta_m$.
We have not made any particular choices, so one easily convinces himself that a natural transformation between simplicial sets induces a topological maps between the geometric realization: it does so obviously on the products and so on the terms of the disjoint union and the equivalence relation is preserved because of the naturality conditions.
Note that $|Hom_\Delta(\bullet, n)| \cong \Delta_n$.
The following holds, more or less easily.

Proposition. *The space $|F|$ admits a CW-complex structure whose n-cell are indexed over the non degenerate n-simplex of F, namely the elements of $F(n)$ which are not images via co-degeneracy maps of elements in $F(< n)$.*

The following is a deep theorem.

Theorem 1 (May). *Given a topological space X and one of its points x_0, the projection map $(|S(X)|, |S(x_0)|) \to (X, x_0)$ induces isomorphism on all homotopy groups.*

For a proof, one can look at May's[2].
A consequence of this is that, when restricted to X a CW-complex, the aformentioned projection map is a homotopy equivalence (by Whitehead's theorem).
Note that, when restricting ourself to some more well-behaved objects, one obtains result such as

Proposition. *If one of $|F|, |G|$ is locally compact then $|F \times G| \cong |F| \times |G|$.*

Moreover, by giving (pointwise) the image of the functor $S_\bullet(?) : Top \to sSet$ the compact-open topology, one obtains a functor $Sing : Top \to sTop$. The geometric realization $|\bullet|$ applied to an element of $sTop$ (so now the simplexes need not be discrete topological spaces) satisfies a nice relation, which we will not prove.

[1]Just to clarify, if not otherwise stated, the results in these pages may work under the hypothesis that R is commutative.

Theorem 2. *By restricting Top to the full subcategory of compactly generated Hausdorff spaces, the pair*

$$|\bullet| : sTop \rightleftharpoons Top : Sing(\bullet)$$

form an adjunction pair.

We are almost finished. Let us now define a functor $\mathcal{C}at \to sSet$, called the *nerve* functor. We will define it on the objects, and then it will be clear how to extend it functorially.
Given a small category $\mathcal{C}$ one can consider the simplicial set $N\mathcal{C}$ by simply exhibiting a set $(N\mathcal{C})_n$ for each n and then expliciting degeneracy and face maps (because they 'generate' the morphisms). We will put $(N\mathcal{C})_n$ to be the set of $n-$tuple of composable morphisms. Then we will define its image under the i-th face map as the $(n-1)$-tuple obtained by composing the i's and $i+1$'th morphisms (or disregarding the last one) and the image under the i'th face maps as the $(n+1)$-uple obtained by inserting an identity morphism at position i.
We are done. Let us restate again the beginnin definition:

Def. *Given a small category $\mathcal{C}$, one calls classifying space of $\mathcal{C}$ the space $|N\mathcal{C}|$.*

This name comes from the following facts: let G be a group regarded as a one object category, then

Proposition. *the classifying space $|NG|$ of the category G is the usual geometric classifying spaces BG.*

In particular BG is a $K(G,1)$: one can obtain $|NG|$ by considering $|N\tilde{G}|$ where $\tilde{G}$ is the category having G as objects and one morphism between every ordered pair of objects. Now G acts (on the right) freely (and cellularly!) on $|N\tilde{G}|$ and we can consider $|N\tilde{G}|/G$.
A little unraveling of what happens shows that $|N\tilde{G}|/G \cong |NG|$ and the quotient map is a covering map.
From now on, we will call $EG := |N\tilde{G}|$ and $BG := |NG|$.

Proposition. *The space EG is contractible.*

The proof is very easy. With the following fact it becomes easier.

Proposition. *A natural transformation $F \Rightarrow G$ induces an homotopy between the maps BF and BG.*

It follows (we won't state much) that categories with an initial or terminal objects (as well as filtering categories) admits a contractible classifying space. Then EG is contractible and hence BG is a $K(G,1)$.
We will stop here from now.

Quillen's Plus construction

The previous section was not so useful for this one, but it will for the definition of Quillen's Q construction. It is still a very generic and popular notion in algebraic topology, nevertheless useful.
Let us fix a ring R and consider the group $GL(R)$ as a topological group by giving it the discrete topology. We now know that there exists a space, its classifying space, which admits as non trivial homotopy groups simply the fundamental group which is then $GL(R)$.
This group is what is usually denoted as $BGL(R)$ which serves as an Eilenberg-Maclane space

$K(GL(R),1)$. This is in fact - by basic theory (which follows the preceeding section) - unique up to homotopy if one just looks at CW-complexes (so, in some sense, if we were to not give a construction, one could just defined this space by specifing its homotopy groups and hoping that it exists).
We now know something about the fundamental group of $BGL(R)$: it contains an interesting normal subgroup: the one spanned by finitely dimensional elementary matrixes, called $E(R)$. This was precisely what we used to define by hand the group $K_1(R)$.
If our main goal is trying to find a space X_R such that the algebraic K-theory of R is encoded (or maybe even coincides) with the homotopy groups of X_R, then the space $BGL(R)$ may be a good starting point. Unfortunately, the first K_1 group should be a quotient of what we have on our hands right now.
If it were the other way round, then maybe finding a Galois' covering of $BGL(R)$ may have helped.
In this situation we want to **quotient out** a normal (and perfect!!) subgroup of the fundamental group.
A standard way to impose new relations on the fundamental group of a CW-complex (which is also a standard hands-on approach to show existence of Eilenberg-Maclane spaces) is by attaching 2-cells with boundary our desired vanishing elements. This is precisely what our construction will consists of.
We are ready to state the main Theorem of existence (and definition) of the Quillen's plus construction from a path connected space.

Theorem 3 (Quillen)**.** *Fix (X,x) a pointed and path connected topological space and $N \triangleleft \pi_1(X,x)$ a normal* ***perfect*** *subgroup. Then there exists a pointed space (X^+,x^+) called* ***plus construction of*** *X and a pointed map $f : (X,x) \to (X^+,x^+)$ such that*

- $0 \to N \to \pi_1(X,x) \xrightarrow{f_*} \pi_1(X^+,x^+) \to 0$ *is exact;*
- $f_* : H_n(X, f^*L) \to H_n(X^+, L)$ *is an isomorphisms for any local sysem on X^+;*
- *the map f is universal with respect to maps $(X,x) \xrightarrow{g} (Y,y)$ sending N to zero at the level of π_1, that is for each g there exist an h unique up to homotopy such that*

$$\begin{tikzcd} (X,x) \arrow[r, "g"] \arrow[d, "f"] & (Y,y) \\ (X^+,x^+) \arrow[ur, "h"'] & \end{tikzcd}$$

  commute.

Proof. We will present the proof of V. Srinivas.
As already spoiled at the beginning of the section, the space X^+ will be obtained by X attaching small dimensional cells and the map f will in fact be a base-pointed inclusion (so $x^+ \equiv x$).
As a first thing, we should choose loops e_α generating N. In fact we only need to normally generate it, but using more cells won't be a problem (think about abelian π_1). Consider now a representative γ_α for each class and then construct the space X_1 as

$$X \bigcup_{\partial D^2_\alpha = \gamma_\alpha} D^2_\alpha.$$

Now we can consider the following pullback space of covering spaces, where $\tilde{X_1} \to X_1$ is assumed to be universal:

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & \tilde{X_1} \arrow[d] \\ X \arrow[r] & X_1 \end{tikzcd}$$

It is clear that one can obtain $\tilde{X_1}$ by attaching 2-cells to $\hat{X}$: given $\hat{X}$ one can attach an appropriate number of 2-cells in such a way that he covers X_1 but is simply connected. By uniqueness of the universal cover, this is a model of $\tilde{X_1}$ which is therefore obtained via this gluing.
Note that commutativity of this diagram, seen at the level of π_1 is telling precisely that the π_1 of $\hat{X}$ is N.
Now consider the abelian group $H_2(\tilde{X_1}, \hat{X})$: this is free and generated by the 2-cell just mentioned. This follows from the theory of CW complex. Now we see that the 2 cells are acted on by $\pi_1(X, x)/N$ in a free matter: the fundamental group of the base is the group of deck transformation of the universal cover and the action is transitive by monodromy.
Hence, $H_2(\tilde{X_1}, \hat{X})$ is a free $\mathbb{Z}[\pi_1(X, x)/N]$-module on the generators given by the pullback of the new $2-$cells along the map $\tilde{X_1} \to X_1$.
We are now almost ready, we are missing one last construction.
Consider the following diagram

$$\begin{tikzcd} \pi_2(\hat{X}) \arrow[r] \arrow[d] & \pi_2(\tilde{X_1}) \arrow[r] \arrow[d] & \pi_2(\tilde{X_1}, \hat{X}) \arrow[r] \arrow[d] & \pi_1(\hat{X}) \arrow[d] \\ H_2(\hat{X}) \arrow[r] & H_2(\tilde{X_1}) \arrow[r, two heads] & H_2(\tilde{X_1}, \hat{X}) \arrow[r] & H_1(\hat{X}) \end{tikzcd}$$

where the vertical maps are the Hurewicz map.
Then clearly, because the H_1 is obtained by abelianizing the fundamental group (which is perfect), we have $H_1(\hat{X}) = 0$.
By commutativity we have that every element of the relative $H_2(\tilde{X_1}, \hat{X})$ is represented via a map $S^2 \to \tilde{X_1}$.
Now, composing this maps with $\tilde{X_1} \to X_1$ we obtain the data of attaching maps $S^2 \to X_1$. We can use these to glue some 3-dimensional cells $(b_\alpha)_\alpha$ obtaining thus a new space, denoted X^+. This will be the plus construction we wanted.
Clearly we did not alter the fundamental group of $\tilde{X_1}$, so the first part of the theorem follows.
Again by considering the universal cover $\tilde{X^+}$ one obtains

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & \tilde{X_1} \arrow[r] \arrow[d] & \tilde{X^+} \arrow[d] \\ X \arrow[r] & X_1 \arrow[r] & X^+ \end{tikzcd}$$

which, from the long exact sequence of the triple at the top row gives us a relative chain complex

$$0 \to C_3(\tilde{X^+}, \hat{X}) \to C_2(\tilde{X^+}, \hat{X}) \to 0$$

It will turn out that the map between the third and second chains module is in fact an isomorphism, not only of abelian group but of **free** $\pi(X, x)/N$-modules.
This complex is then acyclic, and from there we obtain the second thesis of the theorem: looking at homology with coefficients L means choosing a $\mathbb{Z}[\pi_1(X, x)/N]$-module and tensoring

the chain complex. But now $L \otimes_{\mathbb{Z}[\pi_1(X,x)/N]} C_\bullet(\tilde{X}_1, \hat{X})$ is acyclic too, hence $H_\bullet(\tilde{X^+}, \hat{X}; L) \equiv 0$ from which $H_\bullet(X; i^*L) \cong H_\bullet(X^+; L)$ for each system L.
As V. Srinvas, I will omit the proof of the third part of the theorem. □

Note that there is a natural map from the i'th stable homotopy group of the sphere and the i'th K-theory of the integers.
This is obtained by giving a map

$$BS_\infty^+ \to BGL(\mathbb{Z})$$

and one can do this by looking at element of S_∞ as permutations matrix (which are clearly invertible in the integers).
This - by functoriality - induces $\pi_i(BS_\infty^+) \to \pi_i(BGL(\mathbb{Z})) = K_i(\mathbb{Z})$.
By a theorem of Barrat, Priddy and Quillen, the first topological space is $\mathbb{Z} \times \Omega^\infty \Sigma^\infty S^0$ and by Freudenthal's stability, the homotopy groups of this last term are precisely the stable homotopy groups of S^0.
We will now state a couple of easy results which confirms the well behaviour of our plus construction.

Proposition. *Given a pointed cover $(\hat{X}, \hat{x}) \to (X, x)$ corresponding to the perfect subgroup $N \triangleleft \pi_1(X, x)$ then the universal cover of the plus construction $\tilde{X^+}$ is the plus construction of the cover $\hat{X}$.*

Proposition. *Given two pointed spaces and two perfect normal subgroups of their fundamental groups, there is a plus construction associated with their product: this corresponds, up to homotopy, to the product of the plus construction.*

We will now convince ourselves a bit that this is a coherent definition of the K theory groups.
Call $F(R)$ the homotopy fiber of $BGL(R) \to BGL(R)^+$.

Theorem 4.
- *$F(R)$ is acyclic;*
- *$\pi_1(F(R)) \cong St(R)$, the Steinberg group;*
- *$\pi_1(F(R))$ acts trivially on $\pi_i(F(R))$ for $i \geq 2$.*

Proof. Basically, both proofs follow from the Serre's spectral sequence.
Its scope is a bit far from ours, but the tool itself is not complicated in anyway so I will sketch Srinvas's proofs.
As always, we will work with the universal cover $B\widetilde{GL(R)}^+$ and the pullback cover $B\hat{GL}(R)$. They form a fibration and coherently their homotopy fiber is $F(R)$.
We can then use them - which now are simply connected - to calculate the homology of $F(R)$.
Note, importantly, that $B\hat{GL}(R)$ is a $K(E(R), 1)$, as our construction the functor $\bullet^+$ showed.
The Serre spectral sequence in homology states that there exists a spectral sequence with second page

$$E^2_{p,q} = H_p(B\widetilde{GL(R)}^+, H_q(F(R)))$$

converging to $H_{p+q}(B\hat{GL}(R))$, where fortunately the local system is the constant sheaf because of simply-connectedness of $B\widetilde{GL(R)}^+$.
As Srinvasa says, the edge morphism $H_n(B\hat{GL}(R)) \to E^\infty_{n,0} \to E^2_{n,0} = H_n(B\widetilde{GL(R)}^+)$ is the map induced by the inclusion of spaces and is therefore an isomorphism by the most important

property of the plus construction.
That is good news: we just proved that convergence of this spectral sequence can be read on the line $q = 0$ and all other terms of the infinity page are zero (so we don't have any extension problems to solve).
Looking at the first time that $H_q(F(R))$ is non zero, we see that on the second page E^2 all the positive rows before the q'th are zero. That means that $E^2_{0,q}$ has no business in getting killed by any maps (which in any case should happen between the $q+1$'th page and $q+2$'th page) and henche $E^2_{0,q} = E^\infty_{0,q}$. But then we know that the infinity page is concentrated on row zero, which tells us that the first column of E^2 is made of zero.
It contains, however, chains with coefficients in $H_q(F(R))$, thus proving that all homology groups of $F(R)$ are zero.
Now, to prove the second and third statement of the theorem, we will use the existence of a spectral sequence such that

$$E^2_{p,q} = H_p(\overbrace{\pi_1(F(R))}^{G}, H_q(\widetilde{F(R)})) \Rightarrow H_{p+q}(F(R)).$$

The power comes from the fact that $F(R)$ is acyclic: everything in the spectral sequence (except at point $0,0$) needs to get killed.
Because the H_1 of the universal cover is trivial, both the $E^2_{1,0}, E^2_{2,0}$ needs to be zero (they have to be killed but there is no one to kill them).
Similarly. $E^2_{3,0} \to E^3_{0,2} \cong E^2_{0,2}$ should be an isomorphism. One obtains thus

$$H_1(G) = H_2(G) = 0, H_3(G) = H_0(G, H_2(\widetilde{F(R)})).$$

Now we can consider the **homotopy exact** sequence

$$F(R) \to BG\hat{L}(R) \to B\widetilde{GL(R)}^+$$

and note that it induces isomorphisms

$$\pi_{i+1}(BGL(R)^+) \cong \pi_i(F(R)), i \geq 2$$

as well as an exact sequence

$$0 \to \pi_2(BGL(R)^+) \to G \to E(R) \to 0.$$

Now, not going into much detail, this extension is a central extension of $E(R)$ with trivial H_1, H_2. By proposition 1.10 in Srinvas, this completely characterizes the extension we are looking at and because $0 \to K_2(R) \to St(R) \to E(R) \to 0$ satisfies the same thing, one obtains the theorem as well as the following very important corollaries. □

Proposition.

$$G \cong St(R), \quad \pi_i(BGL(R)^+) \cong K_i(R) \textit{ for } i = 1, 2$$

and

$$\pi_3(BGL(R)^+) \cong H_3(St(R)).$$

All the preceeding results are interesting: they give a motivation for why the definition $K_i(R) = \pi_i(BGL(R)^+)$ may be good.
They even give us long exact sequence of K-theory groups: this simply follows from the long

exact sequence of homotopy group applied to the short exact sequence of fibration of Eilenberg-Maclane spaces (to which one applies the plus construction with the due attentions).
We even obtain for free a morphism

$$K_i(R) \to H_i(BGL(R)^+)$$

coming from the Hurewicz map.
Now we can state some results without paying much attention to the proofs.

Theorem 5 (Quillen's computation of K-theory of finite fields.)**.** *The following holds, where p is assumed to be a prime:*

- $K_0(\mathbb{F}_p) = \mathbb{Z}$;
- $K_{2i}(\mathbb{F}_p) = 0$ *for* $i > 0$;
- $K_{2i-1}(\mathbb{F}_p) \cong \mathbb{Z}/(p^i - 1)\mathbb{Z}$ *for* $i > 0$.

According to Srinvasa, Quillen's proof of this calculation was indeed what motivated him for the plus construction.

To conclude. We construct **one** of the different products in K-theory.

Proposition. *The map which assigns* $(M, N) \in GL(R) \times GL(R)$ *the matrix obtained by putting M in the (odd, odd) positions and N in the (even, even) positions and zero elsewhere, is a group homomorphism.*

Using this operation and choosing a homotopy inverse of

$$B(GL(R) \times GL(R))^+ \to BGL(R)^+ \times BGL(R)^+$$

one can obtain a map

$$BHL(R)^+ \times BGL(R)^+ \to B(GL(R) \times GL(R))^+ \to BGL(R)^+$$

which indeed induces a product on the plus construction.

Proposition. *Denoting the aforementioned composition as* $+$, *it turns out that* $(BGL(R)^+, +)$ *is a homotopy commutative and associative connected* $H-space$, *hence a commutative* $H-group$.

References

[1] V. Srinivas. *Algebraic L-Theory*. Springer, 1993.

[2] MAY, J. P. *Simplicial Objects in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.