Exact Categories and Quillen's Q construction

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Let us recall the definition of the K_0 group of a ring R. There are actually two of them (which are equivalent):

1. One can give the set $P(R)$ of isomorphism classes of finitely generated projective R-modules a structure of a monoid by direct sums. Then $K_0(R)$ is defined to be the group completion of this monoid.

2. Alternatively, one can consider the free abelian group generated by $P(R)$ and then quotient out by the subgroup generated by the relations $[M] - [N] - [P]$ for all short exact sequences $0 \to N \to M \to P \to 0$ of finitely generated projective modules.

Indeed, short exact sequences of projective modules always split, so these two definitions are more or less the same thing (easy to write down an isomorphism between them). But if we think about it carefully, these two definitions actually rely on different structures of the category of finitely generated R-modules $\mathcal{P}(R)$. We say that the first uses the symmetric monoidal structure $- \oplus -$ on $\mathcal{P}(R)$ while the second uses the exact structure on it. Today we will focus on the latter perspective.

Exact categories were introduced by Quillen in 1972 for the purpose of his second definition of higher algebraic K-theory: the Q construction.

Definition. *An exact category is an additive category $\mathcal{E}$ endowed with a class C of diagrams (called short exact sequences, ses) of the form*

$$M' \to M \to M''$$

satisfying:

1. *C is closed under isomorphisms.*

2. *C contains all diagrams of the form $M \to M \oplus N \to N$ with canonical arrows (split exact sequences).*

We call morphisms that occur as the first (resp. second) arrow in a ses admissible monomorphisms (resp. epimorphisms). Moreover,

3. *Pushouts (resp. pullbacks) of admissible monomorphisms (resp. epimorphisms) always exist and remain admissibly monic (resp. epic).*

4. *Admissible monomorphisms (resp. epimorphisms) are closed under composition.*

5. *Admissible monomorphisms (resp. epimorphisms) are kernels (resp. cokernels) of their corresponding admissible epimorphisms (resp. monomorphisms).*

It is perhaps interesting to mention that there was an extra condition in the original definition by Quillen, but it was later proven to be redundant by Professor Bernhard Keller in 1990.

Such generality is not useful for us, as all the additive categories that we will encounter appears as subcategories of abelian categories. Notice that abelian categories, with their standard notion of ses, form exact categories. So here's the real definition (for us):

Definition. *An exact category $\mathcal{E}$ is a full additive subcategory of an abelian category $\mathcal{A}$ that is closed under extensions, that is, if we have a ses in $\mathcal{A}$:*

$$0 \to M' \to M \to M'' \to 0$$

such that M' and M'' belong to $\mathcal{E}$, then M belongs to $\mathcal{E}$.

It is not hard to prove that such a category, endowed with the notion of ses inherited by $\mathcal{A}$, satisfies the axioms given by the first definition. One can even prove that these two definitions are equivalent: every exact category in the sense of the first definition arises as a subcategory of an abelian category, by Yoneda-ly embedding it into the category of left exact functors from it to the category of abelian groups Ab. (Quillen-Gabriel embedding theorem).

Examples of exact categories include the full subcategories of the category of R-modules of

- finitely generated projective modules $\mathcal{P}(R)$,
- finitely generated modules $\mathcal{M}(R)$,
- (finitely generated) torsion-free modules,
- (finitely generated) torsion modules.

In the definition for us, an admissible monomorphism $M \to N$ is just a morphism in $\mathcal{E}$ such that in $\mathcal{A}$ it is monic and its cokernel lies in $\mathcal{E}$. We denote them by $M \rightarrowtail N$. Similar thing for admissible epimorphisms, we denote them by $M \twoheadrightarrow N$.

A morphism of exact categories, or an exact functor, is an additive functor that preserves ses. Thus we also have the category of exact categories. Given an exact category $\mathcal{E}$, one can define $K_0(\mathcal{E})$ exactly by the way that we defined $K_0(R) = K_0(\mathcal{P}(R))$ of a ring R. This construction is obviously functorial.

Now let us try to generalize this to higher degrees. We will construct a new category $Q\mathcal{E}$ such that the homotopy groups of its classifying space give the

K-groups of $\mathcal{E}$. $Q\mathcal{E}$ has the same objects as $\mathcal{E}$. A morphism $f : X \to Y$ in $Q\mathcal{E}$ is a (isomorphism class of) diagram of the form $X \twoheadleftarrow Z \rightarrowtail Y$. Here we identify two such diagrams if they fit in a commutative diagram:

$$\begin{tikzcd}
X \arrow[d, equal] & Z \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\wr"] & Y \arrow[d, equal] \\
X & Z' \arrow[l, two heads] \arrow[r, tail] & Y
\end{tikzcd}$$

where the middle vertical arrow is an isomorphism. This identification is needed to make $Q\mathcal{E}$ a locally small category.

Composition of morphisms are defined as follows. Given two morphisms $X \twoheadleftarrow Z \rightarrowtail Y$ and $Y \twoheadleftarrow V \rightarrowtail T$, consider the diagram

$$\begin{tikzcd}
X & & \\
Z \arrow[u, two heads] \arrow[r, tail] & Y & \\
Z \times_Y V \arrow[u, two heads] \arrow[r, tail] \arrow[ur, phantom, "\square"] & V \arrow[u, two heads] \arrow[r, tail] & T
\end{tikzcd}$$

where the square is a Cartesian square. We claim that the canonical compositions $X \leftarrow Z \times_Y V \to T$ are admissible, hence give a morphism in $Q\mathcal{E}$. To see this, one first checks that $\ker(Z \times_Y V \to Z) \simeq \ker(V \twoheadrightarrow Y)$ and $\operatorname{coker}(Z \times_Y V \to V) \simeq \operatorname{coker}(Z \rightarrowtail Y)$. The first is easy since kernels commute with pullbacks, and the second follows either from a Freyd-Mitchell argument or by verifying that the square is also cocartesian (as $V \twoheadrightarrow Y$ is epic). This proves that the left and bottom arrows in the square are admissible. One concludes using the fact that admissible monomorphisms (resp. epimorphisms) are preserved under composition (since $\mathcal{E}$ is exact). Playing with Cartesian squares in the following diagram shows that the composition is associative:

$$\begin{tikzcd}
X & & & \\
Z \arrow[u, two heads] \arrow[r, tail] & Y & & \\
Z \times_Y V \arrow[u, two heads] \arrow[r, tail] \arrow[ur, phantom, "\square"] & V \arrow[u, two heads] \arrow[r, tail] & T & \\
Z \times_Y V \times_V V \times_T U \arrow[u, two heads] \arrow[r, tail] \arrow[ur, phantom, "\square"] & V \times_T U \arrow[u, two heads] \arrow[r, tail] \arrow[ur, phantom, "\square"] & U \arrow[u, two heads] \arrow[r, tail] & S
\end{tikzcd}$$

Now we have a category $Q\mathcal{E}$. It has a distinguished object 0 (not a zero object, though).

Theorem. *There is an isomorphism $K_0(\mathcal{E}) \simeq \pi_1(BQ\mathcal{E}, 0)$, functorial in $\mathcal{E}$.*

Let us just mention how the homomorphism $K_0(\mathcal{E}) \to \pi_1(BQ\mathcal{E}, 0)$ is constructed. Given any object $M \in \mathcal{E}$, there are always two canonical morphisms from 0 to M in $Q\mathcal{E}$: $0 \twoheadleftarrow M = M$ and $0 = 0 \rightarrowtail M$. Composing one with the inverse of the other gives a loop based at 0 in $BQ\mathcal{E}$. Given a ses $0 \to M' \to M \to M'' \to 0$, examining the relations among the morphisms between 0,M,M' and M'' shows that this homomorphism is well-defined (Example 4.10, Srinivas).

Definition. *The K-groups of $\mathcal{E}$ are defined to be*

$$K_i(\mathcal{E}) := \pi_{i+1}(BQ\mathcal{E}, 0) = \pi_i(\Omega BQ\mathcal{E}, 0), i \geq 0.$$

The K-groups of a ring R are defined to be those of the exact category $\mathcal{P}(R)$.

The space $\Omega BQ\mathcal{E}$ is called the K-theory space of $\mathcal{E}$, $K(\mathcal{E})$. Waldhausen has shown that it is possible to deloop it infinitely many times, making it into an infinite loop space, aka a connective spectrum. Notice that contrary to the + construction, the Q construction gives the correct definition of K_0. However, the + construction has the advantage of being easier to check that it coincides with the classical definitions of K_1, K_2.

For the rest of the time we will talk about some fundamental theorems in K-theory, namely the theorems of additivity, resolution, dévissage and localization. We apologize but do not regret for not proving any of them, as they rely heavily on the machinery of simplicial homotopy theory. In fact, it was the systematic study of the Q construction and K-theory that motivated Quillen to discover his Theorem A and Theorem B, which are pure statements about classifying spaces of categories.

Theorem (Additivity)**.** *Assume we have a ses $0 \to F' \to F \to F'' \to 0$ of exact functors between exact categories $\mathcal{C} \to \mathcal{D}$ (which means the sequence is exact pointwise). Then $F_* = F'_* + F''_* : K_i(\mathcal{C}) \to K_i(\mathcal{D})$ for all i.*

For $i = 0$, this theorem is more or less tautological. We see that properties that K_0 satisfy generalize to higher degrees.

Theorem (Resolution)**.** *Let $\mathcal{P}$ be a full additive subcategory of an exact category $\mathcal{E}$. Assume that $\mathcal{P}$ satisfies:*

1. *$\mathcal{P}$ is closed under extensions,*
2. *$\mathcal{P}$ is closed under kernels of admissible epimorphisms,* i.e.*, if $0 \to M' \to M \to M'' \to 0$ is a ses in $\mathcal{E}$ with $M, M'' \in \mathcal{P}$, then $M' \in \mathcal{P}$,*
3. *every $M \in \mathcal{E}$ admits a finite resolution by $\mathcal{P}$,* i.e.*, a natural number $n \in \mathbb{N}$ and a long exact sequence*

$$0 \to P_n \to P_{n-1} \to \cdots P_0 \to M \to 0$$

with $P_i \in \mathcal{P}$, $0 \leq i \leq n$.

Then $\mathcal{P}$ is canonically an exact category, and $K(\mathcal{P}) \to K(\mathcal{E})$ is a homotopy equivalence. In particular, $K_i(\mathcal{P}) = K_i(\mathcal{E})$ for all $i \in \mathbb{N}$.

This theorem is easy to see at the level of K_0: a finite resolution of M yields an equality $[M] = \Sigma_{i=0}^{n}(-1)^i[P_i]$, which implies $K(\mathcal{E})$ is generated by the image of $K(\mathcal{P})$. Injectivity is obvious.

Given a ring R, we may also consider its G-theory, that is, $G(R) := K(\mathcal{M}(R))$, $G_i(R) := \pi_i(G(R), 0)$.

Corollary. *Assume R is a ring such that every finitely generated module admits a finite resolution by finitely generated projective modules (* e.g. *a regular commutative ring with finite Krull dimension), then $G(R) \simeq K(R)$. In particular, $G_i(R) \simeq K_i(R)$ for all i.*

Theorem (Dévissage)**.** *Let $\mathcal{B}$ be a full abelian subcategory of an abelian category $\mathcal{A}$ such that $\mathcal{B}$ is closed under finite limits and colimits. Assume every $M \in \mathcal{A}$ admits a finite filtration in $\mathcal{A}$: $0 = M_n \subset M_{n-1} \subset \cdots M_0 = M$ such that $M_i/M_{i+1} \in \mathcal{B}$ for all $0 \leq i < n$. Then $K(\mathcal{B}) \to K(\mathcal{A})$ is a homotopy equivalence. In particular, $K_i(\mathcal{B}) \simeq K_i(\mathcal{A})$ for all i.*

Again, this theorem is clear at the level of K_0. One can think of this theorem as a way to deal with nilpotency. For example, every $\mathbb{Z}/p^n\mathbb{Z}$-module admits a canonical filtration by $\mathbb{Z}/p\mathbb{Z}$-modules. Thus we have $G(\mathbb{Z}/p^n\mathbb{Z}) \simeq G(\mathbb{Z}/p\mathbb{Z}) \simeq K(\mathbb{Z}/p\mathbb{Z})$ (using the previous corollary).

Let $\mathcal{A}$ be an essentially small abelian category.

Definition. *A Serre subcategory of $\mathcal{A}$ is a full abelian subcategory $\mathcal{B}$ that is closed under finite limits, colimits and extensions.*

Remark that $\mathcal{B}$ is canonically an exact category (as it is closed under extensions).

Given a Serre subcategory $\mathcal{B}$, one may construct the corresponding quotient category $\mathcal{A}/\mathcal{B}$ in the following way: $\mathcal{A}/\mathcal{B}$ has the same objects as $\mathcal{A}$, Hom sets in $\mathcal{A}/\mathcal{B}$ are defined as

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) := \varinjlim_{(M', N')} \mathrm{Hom}_{\mathcal{A}}(M', N/N')$$

where (M', N') runs through the objects in $\mathcal{A}$ such that $M' \subset M$, $N' \subset N$ and $M/M', N' \in \mathcal{B}$. This index set can be endowed with a partial order: $(M', N') \leq (M'', N'') \iff M' \supset M'', N' \subset N''$. Moreover, this partial order is directed as $\mathcal{B}$ is closed under pullbacks and pushouts. Notice that $(M, 0)$ is an initial object, giving rise to a canonical map $\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N)$, so that we have a canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$. $\mathcal{A}/\mathcal{B}$ can be given a structure of an abelian category such that this functor is exact (c.f. Srinivas, Appendix B). This functor has the universal property that it is initial among those that are exact and send $\mathcal{B}$ to 0. In this way, we have developed a certain notion of fibrations in the category of abelian categories and exact functors. The localization theorem states that this notion of "fibrations" behaves well under K-theory.

Theorem (Localization)**.** *Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$. Then applying the K-theory functor on the sequence of exact functors $\mathcal{B} \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ gives a homotopy fibration*

$$K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B})$$

which induces a long exact sequence of abelian groups:

$$\cdots \to K_{i+1}(\mathcal{A}/\mathcal{B}) \to K_i(\mathcal{B}) \to K_i(\mathcal{A}) \to K_i(\mathcal{A}/\mathcal{B}) \to K_{i-1}(\mathcal{B}) \to \cdots$$
$$\cdots \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$$

Let's see an example.

Propostion. *Let A be a Dedekind domain, $K = \mathrm{Frac}(A)$. Denote its set of non-zero prime ideals by $\mathfrak{P}$. Then there is a long exact sequence:*

$$\cdots \to K_{i+1}(K) \to \bigoplus_{\mathfrak{p}\in\mathfrak{P}} K_i(A/\mathfrak{p}) \to K_i(A) \to K_i(K) \to \cdots$$
$$\cdots \to \bigoplus_{\mathfrak{p}\in\mathfrak{P}} K_0(A/\mathfrak{p}) \to K_0(A) \to K_0(K) \to 0$$

Proof. As Dedekind domains and fields satisfy the condition in the corollary of the resolution theorem, it suffices to prove the same result for G-groups.

Set $\mathcal{A} = \mathcal{M}(A)$, $\mathcal{B}$ the full subcategory of finitely generated torsion A-modules, then $\mathcal{B}$ is a Serre subcategory. Apply the localization theorem. The middle term is $K_i(\mathcal{A}) = G_i(A)$. We are left to prove the following two assertions: $K_i(\mathcal{B}) = \bigoplus_{\mathfrak{p}} G_i(A/\mathfrak{p})$ and $K_i(\mathcal{A}/\mathcal{B}) = G_i(K)$.

Let $\mathcal{S} \subset \mathcal{B}$ be the full abelian subcategory of semi-simple modules, that is, those that can be written as a finite direct sum of $A/\mathfrak{p}$. Then the structure theorem of finitely generated modules over Dedekind domains tells us that we may apply the dévissage theorem to $\mathcal{S} \subset \mathcal{B}$. Therefore, we have $K(\mathcal{S}) = K(\mathcal{B})$ up to homotopy equivalence. The structure of the category $\mathcal{S}$ is not hard to understand. It contains $\mathcal{M}(A/\mathfrak{p})$ as full abelian subcategory for each $\mathfrak{p} \in \mathfrak{P}$, moreover, it is a filtered colimit of finite products of these subcategories:

$$\mathcal{S} = \varinjlim_{I\subset\mathfrak{P}\text{ finite}} \prod_{\mathfrak{p}\in I} \mathcal{M}(A/\mathfrak{p})$$

It is a standard fact that the functors Q, nerve N, geometric realization $|\cdot|$ and homotopy groups π_i commute with finite products and filtered colimits (maybe for Q this is not standard but it is not hard to check by hand). Therefore, as K_i is the composition of these functors, we have

$$K_i(\mathcal{S}) = K_i\left(\varinjlim_{I\subset\mathfrak{P}\text{ finite}} \prod_{\mathfrak{p}\in I} \mathcal{M}(A/\mathfrak{p})\right) = \varinjlim_{I\subset\mathfrak{P}\text{ finite}} \prod_{\mathfrak{p}\in I} G_i(A/\mathfrak{p}) = \bigoplus_{\mathfrak{p}\in\mathfrak{P}} G_i(A/\mathfrak{p})$$

which proves the first assertion.

We claim that for $M, N \in \mathcal{A}/\mathcal{B}$, $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M,N) \simeq \mathrm{Hom}_K(M\otimes_A K, N\otimes_A K)$. This implies that $\mathcal{A}/\mathcal{B}$ is equivalent to $\mathcal{M}(K)$, which proves the second assertion (by taking K-groups). By definition, the Hom group in $\mathcal{A}/\mathcal{B}$ is calculated by a filtered colimit, which we may decompose into two:

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M,N) = \varinjlim_{M'} \varinjlim_{N'} \mathrm{Hom}_{\mathcal{A}}(M', N/N').$$

The colimit indexed by N' is easy to calculate, since there is a final object, namely N_{tor}. Set $N_{\mathrm{free}} := N/N_{\mathrm{tor}} \simeq A^{\mathrm{rk}(N)}$, where $\mathrm{rk}(N) = \dim_K(N \otimes_A K)$. For the colimit indexed by M', let $e_1, ..., e_r$ be elements of M that become a K-basis after tensoring with K, $r = \mathrm{rk}(M)$. Then the modules of the form $aAe_1 \oplus \cdots \oplus aAe_r$, $a \in A \setminus \{0\}$ are cofinal in the index category of $\{M'\}$. Hence

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M,N) &\simeq \varinjlim_{a\in A\setminus\{0\}} \mathrm{Hom}_{\mathcal{A}}(aAe_1 \oplus \cdots \oplus aAe_r, N_{\mathrm{free}}) \\
&\simeq \bigoplus_{i=1}^{r} \varinjlim_{a\in A\setminus\{0\}} \mathrm{Hom}_{\mathcal{A}}(aAe_i, A^{\mathrm{rk}(N)}) \\
&\simeq \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{\mathrm{rk}(N)} \varinjlim_{a\in A\setminus\{0\}} aA \\
&\simeq \bigoplus_{i=1}^{\mathrm{rk}(M)} \bigoplus_{j=1}^{\mathrm{rk}(N)} K \\
&\simeq \mathrm{Hom}_K(M \otimes_A K, N \otimes_A K).
\end{aligned}$$

□

Let's see what this sequence tells us in low degrees. The connecting homomorphisms can be descriped explicitly:

$$K^* \simeq K_1(K) \to \oplus_{\mathfrak{p}} K_0(A/\mathfrak{p}) \simeq \oplus_{\mathfrak{p}} \mathbb{Z}$$

identifies with the divisor map, and

$$K_2(K) \to \oplus_{\mathfrak{p}} K_1(A/\mathfrak{p}) \simeq \oplus_{\mathfrak{p}} (A/\mathfrak{p})^*$$

componentwise sends the Matsumoto symbol $\{a, b\}$ to the tame symbol $T_v(a,b) := (-1)^{v(a)v(b)} a^{v(b)}/b^{v(a)} \mod \mathfrak{p}$, where v is the valuation associated to $\mathfrak{p}$.

Therefore the sequences

$$0 \to \mathrm{Cl}(A) \simeq \mathrm{coker}(K^* \xrightarrow{\mathrm{div}} \oplus_{\mathfrak{p}}\mathbb{Z}) \to K_0(A) \to K_0(K) \simeq \mathbb{Z} \to 0$$

$$0 \to \mathrm{coker}(K_2(K) \to \oplus_{\mathfrak{p}}(A/\mathfrak{p})^*) \to K_1(K) \to \ker(K^* \xrightarrow{\mathrm{div}} \oplus_{\mathfrak{p}}\mathbb{Z}) \simeq A^* \to 0$$

are exact, from which we deduce

$$K_0(A) \simeq \mathbb{Z} \oplus \mathrm{Cl}(A)$$

and

$$K_1(A) \simeq A^* \iff K_2(K) \to \oplus_{\mathfrak{p}}(A/\mathfrak{p})^* \text{ is surjective.}$$

For rings of integers of number fields, the last statement holds. This is the ultimate goal of the book *Introduction to Algebraic K-theory* by Milnor. It is related to class field theory.