1 Fundamental Theorem of K-theory

Theorem 1.1
(Srinivas Thm (5.2.)) Let A be Noetherian. Then there are natural isomorphisms for all $i \geq 0$:

i) $G_i(A) \cong G_i(A[t])$ incuded by change of rings

ii) $G_i(A[t,t^{-1}]) \cong G_i(A) \oplus G_{i-1}(A)$ where for $i = 0$, we set $G_{-1}(A) = 0$

Let $B = A[t_1, \dots t_n]$ and let $\mathcal{M}gr_+B$ be the category of positively graded finite B-modules. For any $N \in \mathcal{M}gr_+B$; we can write $N = \bigoplus_{p \geq 0} N_p$ with N_p are finite A-modules. For $p > 0$; we let $N(-p) \in \mathcal{M}gr_+B$ to be the graded module with the grading:

$$N(-p)_m = \begin{cases} N_{m-p} & \text{if } m \geq p \\ 0 & \text{if } m < p \end{cases}$$

The operation $N \to N(-1)$ gives an exact functor from $\mathcal{M}gr_+B$ to itself; thus a homomorphism $T : K_i(\mathcal{M}gr_+B) \to K_i(\mathcal{M}gr_+B)$. This gives $K_i(\mathcal{M}gr_+B)$ a $\mathbb{Z}[t]$ structure where t acts on $K_i(\mathcal{M}gr_+B)$ by T. The base change map $B \otimes_A - : \mathcal{M}(A) \to \mathcal{M}gr_+B$ is exact (as B is trivially flat over A). Therefore it induces a homomorphism: $G_i(A) \to K_i(\mathcal{M}gr_+B)$ and thus induces a map $\psi : G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \to K_i(\mathcal{M}gr_+B)$. Explicitly; for $x \otimes t^n$; $\psi(x \otimes t^n)$ is induced by the functor:

$$B(-n) \otimes_A - : \mathcal{M}(A) \to \mathcal{M}gr_+B$$

We have a proposition whose proof can be found in Srinivas (5.4):

Proposition 1.2
The map $\psi : G_i(A) \otimes_Z \mathbb{Z}[t] \to K_i(\mathcal{M}gr_+B)$ is an isomorphism.

Proof. (Proof of Theorem 2.1.)

i). Let $B = A[t,u]$ and $C = A[t,u,u^{-1}]$. Let $\mathcal{M}gr_+B$ same as above and let $\mathcal{M}grC$ denote the category of finite $\mathbb{Z}$-graded C-modules. Using the canonical equivalence for a multiplicative subset S of A:

$$\text{Mod}_A/\text{Tor}_S(A) \cong \text{Mod}_{S^{-1}A}$$

and applying it on the multiplicative set $\mathcal{S} = (u^i)_{i \geq 0}$; we see that $\mathcal{M}grC$ is naturally equivalent to the quotient of $\mathcal{M}gr_+B$ by the Serre subcategory $\mathcal{M}_\mathcal{S}$ of modules annihilated by a power of u. As any module $M \in \mathcal{M}_\mathcal{S}$ which is annihilated by u is (trivially) in $\mathcal{M}gr_+(A[t])$; we have $\mathcal{M}gr_+(A[t]) \subset \mathcal{M}_\mathcal{S}$ as the full subcategory of modules annihilated by u.

As given any $N \in \mathcal{M}_\mathcal{S}$; one can give a filtration by setting $N_k = \ker\{u^k : N \to N\}$ with given $[n] \in N_k/N_{k-1}$:

$$u^k n = 0 \implies un \in \ker u^k \implies un \in N_{k-1} \implies [un] = 0 \implies N_k/N_{k-1} \in \mathcal{M}gr_+(A[t])$$

Therefore by the Dévissage Theorem;

$$K_i(\mathcal{M}gr_+(A[t])) \cong K_i(\mathcal{M}_\mathcal{S})$$

and we have a natural equivalence of categories $\mathcal{M}grC \xrightarrow{\cong} M(A[x])$ given by $N \mapsto N_0$, where $x = t/u \in C$

By the lthe localization theorem applied to $\mathcal{M}_{\mathcal{S}}$; we have the LES:

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(\mathcal{M}_{\mathcal{S}}) & \longrightarrow & K_i(\mathcal{M}gr_+B) & \longrightarrow & K_i(\mathcal{M}grC) & \longrightarrow & K_{i-1}(\mathcal{M}_{\mathcal{S}}) & \longrightarrow \\
 & \downarrow \cong & & \| & & \downarrow \cong & & \downarrow \cong & \\
\longrightarrow & K_i(\mathcal{M}gr_+(A[t])) & \xrightarrow{\phi} & K_i(\mathcal{M}gr_+B) & \longrightarrow & G_i(A[x]) & \longrightarrow & K_{i-1}(\mathcal{M}gr_+(A[t])) & \longrightarrow
\end{array}$$

By the Proposition 2.2.;

$$K_i(\mathcal{M}gr_+B) \cong G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[y]$$

$$K_i(\mathcal{M}gr_+(A[t])) \cong G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[y]$$

where y is a shift in grading by -1. Therefore ϕ is $\mathbb{Z}[y]$ linear and in order to compute it; we only need to compute it on $G_i(A)$. Given $M \in \text{Mod}(A)$; we have an exact sequence of graded B-modules:

$$0 \to B(-1) \otimes_A M \xrightarrow{u} B \otimes_A M \to A[t] \otimes_A M \to 0$$

where $A[t] = B/uB$. Thus; the functors

$$i : \mathcal{M}(A) \to \mathcal{M}gr_+B$$

$$j(-n) : \mathcal{M}(A) \to \mathcal{M}gr_+B$$

given respectively by $M \mapsto A[t] \otimes_A M$ and $j(-n)M = B(-n) \otimes_A M$ fits into the exact sequence of exact functors:

$$0 \to j(-1) \to j \to i \to 0$$

Hence by the additivity theorem; one has $j_* = j(-1)_* + i_*$ But $j(-1)_* = y \cdot j_*$; thus we have $i_* = (1-y)j_* : G_i(A) \to K_i(\mathcal{M}gr_+B)$ Under the maps of Proposition 2.2.; these fit into the diagram

$$\begin{array}{ccc}
G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[y] & \xrightarrow{\cong_{i_*}} & K_i(\mathcal{M}gr_+(A[t])) \\
{\scriptstyle (1-y)}\downarrow & & \downarrow{\scriptstyle \phi} \\
G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[y] & \xrightarrow[\cong_{j_*}]{} & K_i(\mathcal{M}gr_+B)
\end{array}$$

Therefore identifying ϕ with multiplication by $1-y$. We have $\text{coker}((1-y) : G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[y] \to G_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}[y]) \cong G_i(A)$ via $y \mapsto 1$. Thus $\text{coker}(\phi) \cong G_i(A) \cong G_i(A[x])$ from the long exact sequence. This proves 1).

ii) We let $\mathcal{B} \subset \mathcal{M}(A[t])$ be the Serre subcategory of modules annihilated by a power of t. Again by the same quotienting argument as before; $\mathcal{M}([t])/\mathcal{B}$ is naturally equivalent to $\mathcal{M}(A[t, t^{-1}])$. By the same Devissage argument applied to $\mathcal{M}(A) \subset \mathcal{B}$ of the full subcategory of modules annihilated by t; we see that $K_i(\mathcal{B}) \cong G_i(A)$. Using these identifications and applying the localisation theorem for $(\mathcal{M}(A[t]), \mathcal{B})$ we have a long exact sequence:

$$\cdots \to G_i(A) \to G_i(A[t]) \xrightarrow{f} G_i(A[t, t^{-1}]) \to G_{i-1}(A) \to \cdots$$

We use the isomorphism of i): $g_* : G_i(A) \cong G_i(A[t])$ induced by the change of rings. Similarly one has a map $i_* : G_i(A) \to G_i(A[t, t^{-1}])$ induced again by a change of rings; such that $f \circ g_* = i_*$. The statement of 2) follows if one proves that f is a split inclusion; as g_* is an isomorphism; it is enough to show this for i_*.

Let $\mathcal{M}_1(A[t, t^{-1}]) \subset \mathcal{M}(A[t, t^{-1}])$ be the full subcategory of modules M satisfying $\mathrm{Tor}_1^{A[t,t^{-1}]}(M, A) = 0$ where we identify A as a $A[t, t^{-1}]$-module via $A \cong A[t, t^{-1}]/(t-1)$. Therefore $\mathcal{M}_1(A[t, t^{-1}])$ is exactly the $\mathrm{Tor}_1^{A[t,t^{-1}]}(-, A)$-acyclic objects; by the Corollary 4.7. (Srinivas) of the resolution theorem; one has $K_i(\mathcal{M}_1(A[t, t^{-1}])) \cong G_i(A[t, t^{-1}])$. Furthermore there is an exact(exactness following from vanishing of Tor_1) functor $j : \mathcal{M}_1(A[t, t^{-1}]) \to \mathcal{M}(A)$ given by $M \mapsto A \otimes_{A[t,t^{-1}]} M$. i factors through $\mathcal{M}_1(A[t, t^{-1}])$ and by construction $j \circ i$ is isomorphic to the identity functor. This shows that i_* is split; hence proving the result.

□

Remark 1.3. If R is regular; since $G_i(R) \cong K_i(R)$ via resolution theorem; one has $G_i(R) \cong K_i(R)$ as we have seen before. Therefore this gives splittings in K-theories:

$$K_i(R[t]) \cong K_i(R) \qquad K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R)$$

Remark 1.4. In general for arbitary rings R; the fundamental theorem of Algebraic K-theory or also called as Bass-Heller-Swan decomposition theorem states existence of the following splitting:

$$K_n(R[t, t^{-1}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n^+(R) \oplus NK_n^-(R)$$

where

$$NK_n^{\pm}(R) := \ker(K_n(R[t^{\pm}]) \xrightarrow{t \mapsto 1} K_n(R))$$

Indeed when R is Noetherian regular; $t^{\pm} \mapsto 1$ induces isomorphisms on K-groups; thus Remark 2.3. follows. (proof of BHS can be found in Theorem V8.2. of Weibel K-book)

Examples 1.5.

- Let R be a PID. Then one has $K_0(R[t_1, .., t_n]) \cong K_0(R) \cong \mathbb{Z}$. This proves that every f.g. projective module over $R[t_1, ..t_n]$ is stably free. Although much much stronger results exist (For example Seshadri 1958 proving every f.g. projective module over a polynomial ring with R a PID is free; or even Quillen-Suslin theorem proving this for all R) we can prove this without doing any commutative algebra!

2 Scheme Theory Definitions and Main Theorems

Definition 2.1. (Types of Sheaves $\mathcal{O}_X$-modules we will deal with)

- Intuitively, a **quasi-coherent sheaf** $\mathcal{F}$ on X is an $\mathcal{O}_X$-module such that when restricted to an affine open $U = \mathrm{Spec}\ A$; $\mathcal{F}_{|U} \cong \widetilde{M}$ where $\widetilde{M}$ is the sheaf associated to the A-module M.

  Equivalently; there exists a cover $\{U_\alpha\}_\alpha$ of X such that for each α; there exists I_α and J_α such that the sequence is exact:

$$\mathcal{O}_X^{I_\alpha}|_{U_\alpha} \to \mathcal{O}_X^{J_\alpha}|_{U_\alpha} \to \mathcal{F}|_{U_\alpha} \to 0$$

- A **coherent sheaf** $\mathcal{F}$ on X is an $\mathcal{O}_X$-module such that for any $x \in X$; there exists a $x \in U \subset X$ and a surjection $\mathcal{O}_X^r|_U \to \mathcal{F}|_U$ and if for any $U \subset X$; and any homomorphism $u : \mathcal{O}_X^r|_U \to \mathcal{F}|_U$, the kernel of u is locally finitely generated.
- An $\mathcal{O}_X$-module $\mathcal{F}$ is **locally free** of rank r; if for all $x \in X$; there exists an open neighborhood $U \subset X$ such that $\mathcal{F}|_U \cong \mathcal{O}_X^r|_U$ (Therefore it is a vector bundle)
- A **line bundle** $\mathcal{L}$ on X is a locally free sheaf of rank 1. Similar to the ordinary theory; it satisfies $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$. It is **ample**, if given any coherent sheaf $\mathcal{F}$ on X; there exists $n_0 \geq 0$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by its global sections for $n \geq n_0$.
  If X is of finite type over a Noetherian ring A, then for any ample invertible sheaf L on X, some positive tensor power $L^{\otimes n}$ is of the form $f^*\mathcal{O}_{\mathbb{P}_A^n}(1)$ for some locally closed immersion $f : X \to \mathbb{P}_A^n$

3 K-Theory of Schemes

Let $\mathcal{P}(X) \subset \mathrm{QCoh}(X)$ be the denote the full subcategory of category of locally free sheaves of finite rank. One defines the K-theory of schemes as $K_i(X) := K_i(\mathcal{P}(X))$ If X is furthermore Noetherian; one denotes $\mathcal{M}(X)$ as the abelian category of Coherent sheaves on X, and defines the G-theory (or K'-theory by Quillen) of X as $G_i(X) := K_i(\mathcal{M}(X))$. From now on, all schemes will assumed to be Noetherian and separated.

Example 3.1. When $X = \mathrm{Spec}\ A$ is affine; then a coherent sheaf $\mathcal{F}$ on X is the same as $\mathcal{F} = \widetilde{M}$ for some finitely generated A-module M. Similarly; given a locally free sheaf $\mathcal{G}$ on X is same as $\mathcal{G} = \widetilde{P}$ with P a locally-free (and finitely presented) A-module. But finitely presented locally free modules are same as projective modules.
These two observations give rise to natural equivalences of categories $\mathcal{P}(X) \cong \mathcal{P}(A)$ and $\mathcal{M}(X) \cong \mathcal{M}(A)$; which shows that $K_i(\mathrm{Spec}\ A) \cong K_i(A)$ and $G_i(\mathrm{Spec}\ A) \cong G_i(A)$.

Example 3.2. The inclusion functor $i : \mathcal{P}(X) \hookrightarrow \mathcal{M}(X)$ induces a natural homomorphism $i_* : K_i(X) \to G_i(X)$. If X is regular every coherent sheaf on X is a quotient of a locally free sheaf of finite rank as X is also quasi-compact by Noetherianity, and this gives a finite resolution of locally free sheaves of finite rank. Applying the resolution theorem; we see that i_* is an isomorphism $i_* : K_i(X) \cong G_i(X)$

Remark 3.3. There is an analogue of the fundamental theorem of algebraic K-theory for the K-theory of schemes: (Weibel Theorem 8.3.)

For every quasi-projective scheme X we have canonically split exact sequences for all n, where the splitting of ∂ is by multiplication by t.

$$0 \to K_n(X) \xrightarrow{\Delta} K_n(X[t]) \oplus K_n(X[1/t]) \xrightarrow{\pm} K_n(X[t, 1/t]) \xleftarrow{\partial} K_{n-1}(X) \to 0.$$

in which the splitting of ∂ is given by multiplication by $t \in K_1(\mathbb{Z}[t, t^{-1}])$.

Therefore, the K-theory of schemes in some sense reflect the properties of K-theory of rings.

Given $\mathcal{E} \in \mathcal{P}(X)$; the functor

$$\mathcal{E} \otimes_{\mathcal{O}_X} - : \mathcal{M}(X) \to \mathcal{M}(X)$$

is an exact functor; thus induces a map $[\mathcal{E}]_\star : G_i(X) \to G_i(X)$. By the additivity theorem; an exact sequence of sheaves

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

gives

$$[\mathcal{E}]_\star = [\mathcal{E}']_\star + [\mathcal{E}'']_\star : G_i(X) \to G_i(X)$$

Hence; $\mathcal{E} \mapsto [\mathcal{E}]_\star$ induces a map $K_0(X) \to \mathrm{End}(G_i(X))$ or equivalently a (not necessarily nondegenerate) pairing $K_0 \otimes_{\mathbb{Z}} G_i(X) \to G_i(X)$. This gives $G_i(X)$ and similarly $K_i(X)$ $K_0(X)$-module structures; hence $K_i(X) \to G_i(X)$ is a $K_0(X)$-module homomorphism.

3.1 Functoriality of K-Theory of Schemes

Given a morphism $f : X \to Y$ of (any) schemes; as the pullback of locally free sheaves is exact; this gives a homomorphism: $f^* : K_i(Y) \to K_i(X)$. To define $f^* : G_i(Y) \to G_i(X)$; we require f to be flat between (Noetherian) schemes. Then $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is exact and we have a homomorphism $f^* : G_i(Y) \to G_i(X)$

Let $f : X \to Y$ be a morphism between Noetherian schemes of finite Tor dimension ($\exists N > 0$ s.t. $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = 0 \quad \forall i \geq N \quad \forall \mathcal{F} \in \mathcal{M}(Y)$) (where $\mathcal{O}_X$ is considered as an $f^{-1}\mathcal{O}_Y$-module) and let $\mathcal{M}(Y, f) \subset \mathcal{M}(Y)$ be the full subcategory of sheaves with $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = 0 \ \forall i > 0$. Then one has $\mathcal{P}(Y) \subset \mathcal{M}(Y, f)$. Assuming that every $\mathcal{F} \in \mathcal{M}(Y, f)$ is a quotient of a locally free sheaf; then by the resolution theorem; one has $K_i(\mathcal{M}(Y, f)) \cong G_i(Y)$ for all i. Also as $f^* : \mathcal{M}(Y, f) \to \mathcal{M}(X)$, $\mathcal{F} \mapsto f^*\mathcal{F}$ is an exact functor (which is the reason why we defined $\mathcal{M}(Y, f)$ that way); this gives a map $f^* : K_i(\mathcal{M}(Y, f)) \to G_i(X)$ and under the assumption above; a map $f^* : G_i(Y) \to G_i(X)$. The assumption is satisfied when Y is regular or when Y has a ample line bundle. The relation $g^* \circ f^* = (f \circ g)^* : G_i(Z) \to G_i(Y) \to G_i(X)$ is satisfied when f and g satisfies the assumptions above and Z, Y has ample line bundles.

Now let $f : X \to Y$ be a proper morphism (Which we need for $R^i f_* \mathcal{F} \in \mathcal{M}(Y)$ and $R^i f_* \mathcal{F} = 0$for i large enough for $\mathcal{F} \in \mathcal{M}(X)$ (Hart Thm 8.8)) We let $\mathcal{M}(X, f) \subset \mathcal{M}(X)$ be the full subcategory of sheaves $\mathcal{F}$ with $R^i f_* \mathcal{F} = 0$ for $i > 0$. If further; every sheaf of $\mathcal{F} \in \mathcal{M}(X)$ is a subsheaf of a sheaf $\mathcal{G} \in \mathcal{M}(X, f)$; then by the resolution theorem; we have $K_i(\mathcal{M}(X, f)) \cong K_i(\mathcal{M}(X))$. Therefore under these hypothesis; $f_* : \mathcal{M}(X, f) \to \mathcal{M}(Y)$ is exact; thus induces a map $f_* : G_i(X) \to G_i(Y)$. Again we have $(f \circ g)_* = f_* \circ g_*$ under above assumptions.

The hypothesis of above (every sheaf of $\mathcal{F} \in \mathcal{M}(X)$ is a subsheaf of a sheaf $\mathcal{G} \in \mathcal{M}(X, f)$) is satisfied when f is finite or when X has an ample line bundle. In the second case; let $\mathcal{L}$ be the ample line bundle. As L is ample and f proper; there exists a closed immersion $i : X \to \mathbb{P}_Y^N$ and a positive m satisfying $L^{\otimes m} \cong i^*\mathcal{O}_{\mathbb{P}_Y^N}(1)$ Furthermore; there exists a canonical projection $j : \mathbb{P}_Y^N \to Y$ with $f = j \circ i$. Given any n large enough; one has $n = mk + r$. Since $\mathcal{F}_r := \mathcal{F} \otimes L^{\otimes r}$ is also coherent its enough to prove the result for $n = mk$. ; Then

$$Rf_*(\mathcal{F} \otimes L^{\otimes n}) \cong Rj_* \circ Ri_*(F \otimes i^*\mathcal{O}(n)) \cong Rj_*(Ri_*\mathcal{F} \otimes \mathcal{O}(n)) \cong Rj_*(i_*\mathcal{F} \otimes \mathcal{O}(n))$$

with $Ri_*\mathcal{F} \in \mathcal{M}(\mathbb{P}_Y^n)$ where the last isomorphism follows from the projection formula and the first one implicitly uses Leray Spectral Sequence. Since $Y = Spec(A)$ affine; one has for any sheaf $\mathcal{F}$ on $\mathbb{P}_A^N$; $R^i j_* \mathcal{F} := H^i(\mathbb{P}_A^n, \mathcal{F})$; therefore since for large n; and for $\mathcal{G} = i_*\mathcal{F} \otimes \mathcal{O}(n)$; $H^i(\mathbb{P}_A^N, \mathcal{G})$ vanishes for all $i > 0$ by the Serre Vanishing Theorem; one arrives at $R^i f_* \mathcal{F}(n) = R^i j_*(i_* \mathcal{F} \otimes \mathcal{O}(n)) = 0$ when $Y = Spec(A)$. As the sheaf $R^i f_* \mathcal{F}(n)$ is 0 on all affines; $R^i f_* \mathcal{F}(n) = 0$ on Y. Therefore $\mathcal{F}(n) \in \mathcal{M}(X, f)$ Now take n large enough so that $\mathcal{O}_X(n)$ is also generated by global sections; i.e. we have a surjection $p : \mathcal{O}_X^{\oplus N} \to \mathcal{O}_X(n)$ for some $N > 0$. Furthermore; the kernel of p is a vector bundle; and after dualizing and tensoring with $\mathcal{O}_X(n)$; we get an exact sequence of vector bundles:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^{\oplus N}(n) \to \mathcal{E} \to 0$$

Thus, after tensoring with $\mathcal{F}$; we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{\oplus N}(n) \to \mathcal{E} \otimes \mathcal{F} \to 0$$

. Therefore $\mathcal{F}$ injects into $\mathcal{F}^{\oplus N}(n) \in \mathcal{M}(X, f)$

We will now prove the projection formula for K-theory; which is a functoriality formula that appears in Six-functor formalisms / Representation theory and even in Logic.

Proposition 3.4
(Projection Formula Srinivas 5.12.) Let $f : X \to Y$ be a proper morphism of finite Tor-dimension between schemes supporting ample line bundles (so that f^*, f_* from G-theories are both defined). Then:

i) there is a well-defined map $f_* : K_i(X) \to K_i(Y)$ making the diagram:

$$\begin{array}{ccc} K_i(X) & \longrightarrow & G_i(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_i(Y) & \longrightarrow & G_i(Y) \end{array}$$

commute.

ii) for any $x \in K_0(X)$ and $y \in G_i(Y)$; we have

$$f_*(x) \cdot y = f_*(x \cdot f^* y)$$

iii) for any $x \in K_0(X)$ and $y \in K_i(Y)$ we have

$$f_*(x) \cdot y = f_*(x \cdot f^* y)$$

iv) for any $y \in K_0(Y)$ and $x \in G_i(X)$ we have

$$f_*(f^* y \cdot x) = y \cdot f_* x$$

Where "$\cdot$" is the K_0-module structure on the given K_0-modules. For iv) to hold, one does not need the requirement of an ample line bundle on Y

Proof.
ii) and *iii*) have the exact same proof and the proof of *ii*) applies to *iii*).

proof of iv): Given $\mathcal{E} \in \mathcal{P}(X, f)$ and $\mathcal{F} \in \mathcal{M}(Y)$, one has the projection formula:

$$f_*(f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}) \cong \mathcal{F} \otimes_{\mathcal{O}_Y} f_*\mathcal{E}$$

which is bi-exact with respect to $\mathcal{F}$ and $\mathcal{E}$ when f is proper and X has an ample line bundle. Therefore they are the same as homomorphisms $f_*(f^*y \cdot -) = y \cdot f_*(-) : G_i(X) \to G_i(Y)$.

proof of ii): Let $\mathcal{P}(X, f) = \mathcal{P}(X) \cap \mathcal{M}(X, f)$. By the argument before; for $\mathcal{E} \in \mathcal{P}(X)$; there exists $\mathcal{E}' \in \mathcal{P}(X, f)$ and an admissable monomorphism $\mathcal{E} \to \mathcal{E}'$ in $\mathcal{P}(X)$. This implies $K_i(\mathcal{P}(X, f)) = K_i(\mathcal{P}(X)) = K_i(X)$. Given $\mathcal{E} \in \mathcal{P}(X, f)$; we will show that $f_*\mathcal{E} \in \mathcal{H}(Y)$ which is the full subcategory of $\mathcal{M}(Y)$ consisting of sheaves with finite resolution by locally free sheaves (= finite homological dimension). By the resolution theorem $K_i(\mathcal{H}(Y)) = K_i(Y)$ so this would imply that $f_*\mathcal{E} \in K_i(Y)$. As this property is local; we may assume that $Y = \text{Spec } A$ is affine.

As f has finite Tor dimension; for any affine open $U = \text{Spec } B \subset X$; there exists $N > 0$ such that $\text{Tor}_i^A(B, -) = 0$ for all $i \geq N$ where we may choose N to be independent of U. We take an affine open cover $\{U_i\}_{0 \leq i \leq n}$ of X by a finite number of opens; such that on each U_i; $\mathcal{E}|_{U_i}$ is trivial. As f is proper; it is separated, and this implies that all the intersections of $U_i's$ are affine too and we can compute $H^i(X, \mathcal{E})$ via using its Cech resolution. As $\mathcal{E} \in \mathcal{P}(X, f)$ $0 = R^i f_*\mathcal{E} = H^i(X, \mathcal{E})$ for $i > 0$ and $f_*\mathcal{E} = R^0 f_*\mathcal{E} = H^0(X, \mathcal{E})$ (as sheaves). Therefore, we have a long exact sequence:

$$0 \to H^0(X, \mathcal{E}) \to \bigsqcup_i H^0(U_i, \mathcal{E}) \to \bigsqcup_{i,j} H^0(U_i \cap U_j, \mathcal{E})) \to \cdots$$

By applying Tor L.E.S. over and over; we have $\text{Tor}_i^A(H^0(X, \mathcal{E}), -) = 0$ for $i \geq N$. This proves that $f_*\mathcal{E}$ has finite homological dimension; thus $f_*\mathcal{E} \in \mathcal{H}(Y)$. This proves the first statement.

ii) We want to represent the action of $[f_*\mathcal{E}] \in K_0(Y)$ by a functor. Using that for $\mathcal{E} \in \mathcal{P}(X, f)$; $f_*\mathcal{E} \in \mathcal{H}(Y)$; we can have a finite resolution:

$$0 \to \mathcal{E}_m \to \mathcal{E}_{m-1} \to \cdots \to \mathcal{E}_0 \to f_*\mathcal{E} \to 0$$

with $\mathcal{E}_i \in \mathcal{P}(Y)$. Then by definition of $f_*\mathcal{E}$; given $y \in G_i(Y)$; the module action of $[f_i\mathcal{E}]$ is given by $y \mapsto \sum_{i=0}^n (-1)^i [\mathcal{E}_\rangle] \cdot y$ Let $\mathcal{N} \subset \mathcal{M}(Y, f)$ be the full subcategory of sheaves with $\text{Tor}_i^{\mathcal{O}_Y}(f_*\mathcal{E}, \mathcal{F}) = 0$. Then; tensoring the above resolution with $\mathcal{F}$ gives an exact sequence

$$0 \to \mathcal{E}_m \otimes \mathcal{F} \to \cdots \to \mathcal{E}_0 \otimes \mathcal{F} \to f_*\mathcal{E} \otimes \mathcal{F} \to 0$$

where we interpret each $\mathcal{E}_i \otimes - : \mathcal{N} \to \mathcal{M}(Y)$ as a functor; therefore in the end we have a long exact sequence of functors. Hence the action of $[f_*\mathcal{E}] \in K_0(Y)$ is exactly the functor $f_*\mathcal{E} \otimes - : \mathcal{N} \to \mathcal{M}(Y)$ by the general additivity theorem.

Now using the derived projection formula

$$\mathbf{R} f_*(\mathcal{E} \otimes^L f^*\mathcal{F}) \cong \mathbf{R} f_*\mathcal{E} \otimes^L \mathcal{F}$$

Since $\mathcal{E} \in \mathcal{P}(X, f)$; $\mathbf{R}f_*\mathcal{E} = f_*\mathcal{E}$ and furthermore; as $\mathcal{F} \in \mathcal{N}$; the RHS is not a derived tensor product. Therefore the RHS simplifies to $f_*\mathcal{E} \otimes \mathcal{F}$. Furthermore; as $\mathcal{E} \in \mathcal{P}(X)$; the LHS is also not a derived tensor product; therefore we obtain: $R^i f_*(\mathcal{E} \otimes f^*\mathcal{F}) = f_*\mathcal{E} \otimes \mathcal{F}$ when $i = 0$ and $R^i f_*(\mathcal{E} \otimes f^*\mathcal{F}) = 0$ otherwise. Since both are exact as functors $\mathcal{N} \to \mathcal{M}(Y)$; we have $(f_*x) \cdot y = f_*(x \cdot f^*y)$ for any $y \in G_i(Y)$ and $x = [\mathcal{E}]$ with $\mathcal{E} \in \mathcal{P}(X, f)$. But such classes x generate $K_0(X)$ and both sides are additive in x. Thus this is true for all $x \in K_0(X)$. This completes the proof. □

Now we give a series of results; whose proofs can be found in Srinivas Chapter V.

Let $i : Z \to X$ be a closed subscheme; $j : U \to X$ its open complement. Let I_Z be the sheaf of ideals of Z in $\mathcal{O}_X$. one can identify $\mathcal{M}(Z)$ via the full subcategory of $\mathcal{M}(X)$ consisting of sheaves that are annihilated by I_Z via i_*. By devissage theorem; if I_Z is nilpotent; then $G_i(Z) \cong G_i(X)$. Furthermore, there is a long exact sequence:

$$\cdots \to G_{i+1}(U) \to G_i(Z) \xrightarrow{i_*} G_i(X) \xrightarrow{j_*} G_i(U) \to G_{i-1}(Z)$$

$$\to \cdots \to G_0(X) \to G_0(U) \to 0$$

. This long exact sequence is natural w.r.t. nested closed subschemes: If

$$Z \xrightarrow{i} Z' \xrightarrow{i'} X$$

and

$$X - Z' \xrightarrow{j'} X - Z \xrightarrow{j} X$$

then we have a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & G_{i+1}(X-Z) & \longrightarrow & G_i(Z) & \xrightarrow{(i' \circ i)_*} & G_i(X) & \xrightarrow{j^*} & G_i(X-Z) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle j'^*} & & \downarrow{\scriptstyle i_*} & & \| & & \downarrow{\scriptstyle j'^*} & \\
\cdots \longrightarrow & G_{i+1}(X-Z') & \longrightarrow & G_i(Z') & \xrightarrow{i'_*} & G_i(X) & \xrightarrow{(j \circ j')^*} & G_i(X-Z') & \longrightarrow \cdots
\end{array}$$

When $U, V \subset X$ are subschemes; one can apply the diagram to $Y = X - U$ and $Z = X - V$ to get the Mayer-Vietoris sequence for G-theory:

$$\cdots \to G_{i+1}(U \cap V) \to G_i(U \cup V) \to G_i(U) \oplus G_i(V) \to G_i(U \cap V) \to \cdots$$

Proposition 3.5
(Homotopy Property) Let $f : P \to X$ be a flat map; whose fibers are affine spaces. Then

$$f^* : G_i(X) \to G_i(P)$$

is an isomorphism for all i

We have seen from fundamental theorem of algebraic K-theory and from $G_i(\text{Spec } A) \cong G_i(A)$ that G-theory is $\mathbb{A}^1$-invariant on Noetherian affine schemes; and similarly K-theory is $\mathbb{A}^1$-invariant on regular Noetherian affine schemes. A corollary of the Homotopy property is that this is true for all Noetherian (respectively regular Noetherian) schemes.