K-Theory working group

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Goals for today:
- Recap: projective schemes!
- Prove the homotopy property that was stated last time.
- Compute G-theory of projective bundles.
- Introduce the filtration by codimension of support.

Notations:
- $\mathsf{P}(X)$ is the category of locally free sheaves of finite rank on a scheme X.
- $\mathsf{Coh}\, X$ is the category of coherent sheaves on a scheme X.
- $\mathsf{M}(A)$ is the category of finitely generated modules on a ring A.
- $\mathsf{Mf}\, A$ is the Serre subcategory of $\mathsf{M}(A)$ consisting of finite length objects.
- $\mathsf{Mgr}\, A$ is the category of positively graded finitely generated modules over a graded ring A.
- $\mathsf{Mfgr}\, A$ is the Serre subcategory of $\mathsf{Mgr}\, A$ consisting of finite length objects.

1 Projective schemes

Why projective schemes?

- Basically all classical algebraic geometry is about (quasi-)projective varieties.
- They are the most well behaved schemes you can hope for.
- In many different geometries, there are GAGA-type theorems which say "for projective varieties, algebraic geometry and [whatever geometry] are the same".

For instance the eponymous GAGA (Géométrie Algébrique et Géométrie Analytique) implies that all projective complex analytic varieties are (the analytification of) projective schemes. You can turn some complex analytic questions into algebraic geometry questions and compute things there (often much easier!).

For the remainder of this section, we fix an arbitrary base scheme S and define all notions relative to S. Classical algebraic geometry deals with the case $S = \operatorname{Spec} \bar{k}$ for an algebraically closed field $\bar{k}$.

So what *is* a projective scheme? The shortest way to define this is to say that a morphism $X \to S$ is ***projective*** (or that X is a ***projective S-scheme***) should there exist a closed immersion $X \hookrightarrow \mathbb{P}^r_S$ of S-schemes (which is to say the inclusion must be compatible with the structure maps to S) into the projective space $\mathbb{P}^r_S = \mathbb{P}^r_{\mathbb{Z}} \times S$. For instance, given a family of homogeneous polynomials $f_1 \dots f_n$ in r variables over a ring A, they define an ideal sheaf $\mathscr{I}_f \subseteq \mathscr{O}_{\mathbb{P}^n_A}$ which in turn gives rise to a closed subscheme $V(f_1 \dots f_r)$ of $\mathbb{P}^r_A$. This is the prototypical example of a projective A-scheme, and indeed the only one when A is Noetherian (otherwise you need to take infinite families into account).

Remark 1. Beware! An homogeneous polynomial $f(x_1 \dots x_n)$ does **not** induce a global section of $\mathscr{O}_{\mathbb{P}^n}$. Rather the projective space has a canonical open cover by $n+1$ affine spaces $U_i = \{x_i \neq 0\} \simeq \mathbb{A}^n$, and f induces on each the section

$$f(x_1 \dots x_{i-1}, 1, x_{i+1} \dots x_n) \in \mathscr{O}_{\mathbb{P}^n}(U_i)$$

by evaluating the variable x_i at 1. This is enough to define the ideal sheaf locally on U_i and glue it back to $\mathbb{P}^n_A$.

However, this definition is not always the easiest to work with. Taking inspiration from the Spec construction, we introduce the Proj construction.

1.1 The Proj construction

Definition 2. *Let $R = \bigoplus_{\mathbb{N}} R_n$ a graded ring. An ideal $I \subseteq R$ is said **homogeneous** when it admis a set of homogeneous generators, or equivalently said if it is a graded submodule of R. An homogeneous ideal is **prime** if for any homogeneous a, b such that $ab \in I$, at least one lies in I. There is one prime we care not for, the **irrelevant ideal** $R_+ = \oplus_{\mathbb{N}} R_n$. Then we let*

$$\operatorname{Proj} R = \{\mathfrak{p} \text{ homogeneous prime not containing } R_+\},$$

with the topology having for closed subsets the

$$V(I) = \{\mathfrak{p} : \operatorname{Proj} R \,|\, I \subseteq \mathfrak{p}\}.$$

Denote by $D_+(I)$ the open complement to $V(I)$. Similarly to the Zariski case, the $D_+(f)$ define an open basis for the topology. Finally the space $\operatorname{Proj} R$ is made into a locally ringed space, with structure sheaf defined on the basis by

$$\mathscr{O}(D_+(f)) = (R_f)_0 = \{a/f^n \,|\, a \in R \text{ such that } \deg a = n \deg f\}$$

the degree 0 homogeneous elements of the localisation.

At a glance, this seems much harder! But this definition recovers the examples we know and love.

Example 3. Let $R = A[T_0 \dots T_r]$, graded by letting each T_i have degree 1. Then $\operatorname{Proj} R = \mathbb{P}^r_A$. This can be seen with the gluing construction of the right hand side: $\operatorname{Proj} R$ is covered by $r+1$ open subsets

$$D_+(T_i) = \{\mathfrak{p} : \operatorname{Proj} R \,|\, T_i(\mathfrak{p}) \neq 0\} \simeq \mathbb{A}^r$$

and all the intersections behave as expected. To get an idea of why this is isomorphic to the affine space, send a polynomial $f \in A[T_0 \dots \hat{T}_i \dots T_r]$ to its ***homogeneisation*** $f(T_1/T_i, \dots, T_r/T_i) \in \mathscr{O}(D_+(T_i))$.

Example 4. Let I an homogeneous ideal of R. Then $\operatorname{Proj} R/I$ is a closed subscheme of $\operatorname{Proj} R$, whose underlying subspace is $V(I)$. If R is as in the previous example, this will also coincide with the closed subscheme introduced above remark 1.

Example 5. The scheme Proj R comes together with a very interesting family of line bundles $\mathscr{O}(d)$, defined on the basis by

$$\mathscr{O}(d)(D_+(f)) = (R_f)_d = \{a/f^n \mid a \in R \text{ such that } \deg a - n \deg f = d\}$$

the degree d homogeneous elements of the localisation. For instance $\mathscr{O}(0)$ is nothing but the structure sheaf. This formula actually also holds for the global sections (this isn't hard to prove), so that

$$\Gamma(\operatorname{Proj} R, \mathscr{O}(d)) = R_d,$$

generalising the well-known fact that $\Gamma(\mathbb{P}^r_k) = k$.

Example 6. If E is finite type projective A-module, we can define the ***projective bundle*** $\mathbb{P}(E) = \operatorname{Proj} \operatorname{Sym} E$. It has the property that $f_*\mathscr{O}(n) = E^{\otimes n}$. When $E = A^r$, we recover $\mathbb{P}(E) = \mathbb{P}^r_A$.

Just like for the Spec construction, we may also define a relative version of the Proj construction which on the base scheme S takes in input a graded quasicoherent $\mathscr{O}_S$-algebra $\mathscr{A} = \bigoplus \mathscr{A}_n$, and outputs a scheme Proj $\mathscr{A}$ together with a structure map to S. Just like the affine case, this is a bit of a pain to construct (but not overly hard, we simply need to glue the absolute Proj over all affine opens of S) and I will not do this here.

Example 7. If $\mathcal{E}$ is finite rank locally free $\mathscr{O}_S$-module, we can define the ***projective bundle*** $\mathbb{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E}$. It has the property that $f_*\mathscr{O}(n) = \mathcal{E}^{\otimes n}$. When $\mathcal{E} = \mathscr{O}_S^r$, we recover $\mathbb{P}(\mathcal{E}) = \mathbb{P}^r_S$.

Remark 8. There is a slightly complicated universal property: for any S-scheme $f : T \to S$, scheme morphisms $T \to \operatorname{Proj} \mathscr{A}$ over S are in bijection with equivalence classes of graded algebra maps

$$\psi : f^*\mathscr{A} \to \bigoplus\nolimits_{\mathbb{N}} \mathcal{L}^{\otimes n}$$

of some degree d, in which $\mathcal{L}$ is some invertible sheaf and $\mathscr{A}_d \to \mathcal{L}$ is surjective. Under very mild conditions (that $\mathscr{A}$ be generated by $\mathscr{A}_1$ over $\mathscr{A}_0$), we can drop the variance on d in this property. In the case $\mathscr{A} = \mathscr{O}_S[T_1 \dots T_r]$, this amounts to saying that

$$\operatorname{Hom}_S(T, \mathbb{P}^r) = \big\{ \text{ tuples } \big(\mathcal{L} \text{ invertible sheaf}, f_0 \dots f_r \in \Gamma(\mathcal{L})\big) \,\big|\, f_0 \dots f_r \text{ generate } \mathcal{L} \big\} \,/\sim .$$

Unlike the affine case, we don't say that an S-scheme X is projective whenever it is isomorphic to some Proj $\mathscr{A}$, and would impose some sane conditions on $\mathscr{A}$ that make it workable with. We will restrict to the case $\mathscr{A}_0 = \mathscr{O}_S$, and $\mathscr{A}$ is generated in degree 1, and $\mathscr{A}_1$ is of finite type as an $\mathscr{O}_S$-module. In other words we require that there be a finite family of global sections $t_1, \dots, t_r \in \mathscr{A}_1(S)$ such that the induced morphism of graded $\mathscr{O}_S$-algebras

$$\operatorname{Sym} \mathscr{O}_S^r \to \mathscr{A}$$

be surjective.

Definition 9. *An S-scheme X is* ***projective*** *if it verifies any of the equivalent conditions:*

(a) There is a closed immersion (over S) of X into some $\mathbb{P}^r_S$.

(b) There is a closed immersion (over S) of X into some $\mathbb{P}(\mathcal{E})$.

(c) The scheme X is isomorphic (over S) to some Proj $\mathscr{A}$*, subject to the restriction outlayed just above.*

1.2 Coherent sheaves

On a scheme X, you should have already seen two classes of very interesting $\mathscr{O}_X$-modules:

(1) Algebraic vector bundles (ie. locally free sheaves of finite rank), on which one can read much of the geometry of X, and

(2) Quasi-coherent sheaves, a very flexible superset of the former which forms an abelian category (which enables cohomological method through very streamlined machinery).

In the study of projective schemes, a new intermediate class becomes an extremely important object of study: coherent sheaves, the "good" notion of (locally) finitely presented $\mathscr{O}_S$-modules.

Definition 10. *A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is coherent if*

(a) it is locally finitely generated, which is to say that for $x \in X$ there is an open neighborhood $U \ni x$ and a surjection $\mathscr{O}_U^n \to \mathscr{F}|_U$ for some integer n, and

(b) for any open U and any surjection $u : \mathscr{O}_U^n \to \mathscr{F}|_U$, the kernel of u is itself locally finitely generated.

In particular, as locally the cokernel of a morphism between free sheaves, any coherent sheaf is quasi-coherent. In fact when X is Noetherian the condition simplifies to say that $\mathscr{F}$ is coherent if for any affine open $U \subseteq X$ we have $\mathscr{F}|_U \simeq \tilde{M}$ for some finitely generated module M.

Example 11. Any algebraic vector bundle is coherent.

We denote by Coh X the category of coherent sheaves[1] on a scheme X; it forms a weak Serre subcategory of QCoh X. Unlike for quasi-coherent sheaves, it is not always true that given any morphism $f : Y \to X$ and coherent sheaf $\mathscr{F}$, the pullback $f^*\mathscr{F}$ remains coherent. However the statement is true in many cases of interest, for example should X and Y be locally Noetherian.

Coherent sheaves on projective schemes possess a strikingly beautiful theory, but that story is not to be told in this working group save for scattered bits and pieces.

2 The homotopy property

[1]Last week, this category was called $\mathsf{M}(X)$.

Proposition 12 (Homotopy property). *Let P, X Noetherian schemes, and $f : P \to X$ a flat map all of whose fibers are affine spaces. Then $f^* : G_i(X) \to G_i(P)$ is an isomorphism for all i.*

The usual intuition for flatness is that fibers $P_x \simeq \mathbb{A}^{n_x}_{k(x)}$ vary continuously on X, which between irreducible schemes has some precise corollaries such as that dimension doesn't depend on x. Since we have supposed the fibers to be affine spaces, obviously classified by dimension, they are constant equal to $\mathbb{A}^n$ in some sense. In other words, in homotopic language, the hypotheses translate more or less to "let $P \to X$ a fibration with contractible fiber". It isn't surprising then that $G(P) \simeq G(X)$ for an $\mathbb{A}^1$-homotopy invariant such as G-theory.

Proof [Sri96, 5.17]. We proceed by reduction until we end up at a trivial case. By Noetherian induction, we may replace X with a closed subset $Z \subseteq X$ minimal among those for which $f_Z := f \times_X Z$ fails to verify the conclusion. In other words we can freely assume that f_Z verifies the property for every proper closed subset $Z \subsetneq X$. Remark how we needn't specify the scheme structure on the closed subsets: recall from last time [Sri96, 5.14] that $G_i(Z) = G_i(Z_{\text{red}})$.

For any closed subset/open complement pair $Z \hookrightarrow X \hookleftarrow U$, the localisation sequence [Sri96, 5.15] yields

$$\begin{tikzcd}
\dots \arrow[r] & G_{i+1}(U) \arrow[r] \arrow[d, "f_U^*"] & G_i(Z) \arrow[r] \arrow[d, "f_Z^*"] & G_i(X) \arrow[r] \arrow[d, "f^*"] & G_i(U) \arrow[r] \arrow[d, "f_U^*"] & G_{i-1}(Z) \arrow[r] \arrow[d, "f_Z^*"] & \dots \\
\dots \arrow[r] & G_{i+1}(P_U) \arrow[r] & G_i(P_Z) \arrow[r] & G_i(P) \arrow[r] & G_i(P_U) \arrow[r] & G_{i-1}(P_Z) \arrow[r] & \dots
\end{tikzcd}$$

implies that if the conclusion holds for any two of (Z, X, U), so does it for the third. In particular, applying this two times in the case that $X = Z_1 \cup Z_2$ is not irreducible, it holds for both Z_2 and $Z_2 - (Z_1 \cap Z_2)$ hence for X.

It remains only to deal with the case X irreducible (and reduced). We saw last time [Sri96, 5.16] that the localisation sequence was functorial in $Z \subseteq X$. Thus, taking the (filtered!) colimit of the localisation sequence over the lattice of proper closed subsets, we obtain

$$\begin{tikzcd}
\dots \arrow[r] & \varinjlim G_{i+1}(U) \arrow[r] \arrow[d] & \varinjlim G_i(Z) \arrow[r] \arrow[d, "\cong"] & G_i(X) \arrow[r] \arrow[d] & \varinjlim G_i(U) \arrow[r] \arrow[d] & G_{i-1}(Z) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \varinjlim G_{i+1}(P_U) \arrow[r] & \varinjlim G_i(P_Z) \arrow[r] & G_i(P) \arrow[r] & \varinjlim G_i(P_U) \arrow[r] & G_{i-1}(P_Z) \arrow[r] & \dots
\end{tikzcd}.$$

In order to finish the proof, we need the following

Lemma 13.

- *Let $(X_i)_I$ a cofiltered system of schemes with affine transition maps, and $X = \varprojlim_I X_i$. Then for all q we have $K_q(X) = \varinjlim_I K_q(X_i)$.*
- *If further X and all X_i are Noetherian, and transition maps are flat, then for all q we have $G_q(X) = \varinjlim_I G_q(X_i)$.*

It follows from it that $\varinjlim G_i(U) = G_i(k(X))$ and $\varinjlim G_i(P_U) = G_i(P_{k(X)})$, and we only have to prove the proposition in the case that X is a field. But we already saw that $G_i(\mathbb{A}^n_k) = G_i(k)$ in the fundamental theorem of K-theory [Sri96, 5.2]. $\square$

Proof sketch of the lemma, [Sri96, 5.9]. Although not strictly equal, for any $i \to j \to k$ in I the functors

$$(i \to k)^* : \mathsf{P}(X_k) \to \mathsf{P}(X_i) \quad \text{and} \quad (i \to j)^*(j \to k)^* : \mathsf{P}(X_k) \to \mathsf{P}(X_i)$$

agree up to natural isomorphism, and those natural isomorphisms themselves verify coherence conditions. Therefore, the category $\mathsf{P}(X)$ may be realized as the filtered homotopy colimit of the $\mathsf{P}(X_i)$ in the 2-category of categories. The proof of this lemma refines [Sri96, 3.8], which says that $\pi_{i+1}(B-)$ commutes with filtered 1-colimits, by replacing every $\mathcal{P}(X_i)$ with an explicit equivalent model which kills the higher data (this processus doesn't alter the homotopy colimit, as unlike the 1-colimit it equivalence equivariant). This is technical and uninteresting, we won't do it. $\square$

Proof sketch of lemma [Sri96, 3.8] *in case we didn't do it before.* The nerve functor commutes with filtered 1-colimits because the categories $\Delta[n]$ are compact (finite amount of objects and morphisms). As geometric realization is a left adjoint, it commutes with all colimits. The homotopy groups commute with filtered colimits of CW-complexes, and we are done. $\square$

3 Projective bundles

Let $\mathcal{E}$ a vector bundle of rank r over a Noetherian scheme X. Recall that we can define the associated ***projective bundle*** $\mathbb{P}(\mathcal{E})$ by

$$\mathbb{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E} = \operatorname{Proj} \bigoplus_{\mathbb{N}} \mathcal{E}^{\otimes n},$$

a projective X-scheme $f : \mathbb{P}(\mathcal{E}) \to X$ with the property that $f_*\mathscr{O}(n) = \mathcal{E}^{\otimes n}$ for every n. According to Srinivas, it is "classical" that $K_0(\mathbb{P}(\mathcal{E}))$ is a free $K_0(X)$-module of rank r, a basis being given by the $1, z, \ldots, z^{r-1}$ where z is the class of $\mathscr{O}(-1)$. However the reference he provides, albeit a very good book collecting lecture notes of Manin, only proves this fact under a slew of hypotheses including regularity. In fact it is shown the following more precise formula:

Theorem 14. *Let X a connected regular Noetherian scheme with an ample sheaf, $\mathcal{E}$ a locally free sheaf of rank r on it, and $\mathcal{L} = \mathscr{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then the ring $K_0(\mathbb{P}(\mathcal{E}))$, as a $K_0(X)$-algebra, has the form*

$$K_0(\mathbb{P}(\mathcal{E})) = K_0(X)[T] \Big/ \Big(\sum_{i=0}^{r} (-1)^i \lambda^{r-i}(\mathcal{E}) T^i \Big),$$

where $\lambda^i(\mathcal{E}) = [\bigwedge^i \mathcal{E}]$ (this depends only on the class of $\mathcal{E}$ in K_0!). Moreover this isomorphism is realised by evaluating T at the class $[\mathcal{L}]$.

We admit this, see [Man69, 4.5] for a (very explicit and rather elementary, but pretty long!) proof starring the Koszul resolution as the main character.

Proposition 15. *For each $i \geq 0$, there is an isomorphism of $K_0(\mathbb{P}(\mathcal{E}))$-modules*

$$\begin{aligned} K_0(\mathbb{P}(\mathcal{E})) \otimes_{K_0(X)} G_i(X) &\longrightarrow G_i(\mathbb{P}(\mathcal{E})) \\ (y \otimes x) &\longmapsto y \cdot f^* x. \end{aligned}$$

Proof. As the projectivization behaves well under base change, namely $\mathbb{P}(\mathcal{E}) \times_X T = \mathbb{P}(\mathcal{E}|_T)$, we may apply the same techniques as in the proof of the homotopy property (proposition 12). This reduces the proof to the case $X = \operatorname{Spec} k$ a field, $\mathcal{E}$ an r-dimensional k-vector space, from which follows that

$$A \coloneqq \operatorname{Sym} \mathcal{E} = k[T_0 \dots T_{r-1}]$$

and the projective bundle $\mathbb{P}(\mathcal{E}) = \mathbb{P}^{r-1}_k$ becomes the ordinary projective space. There is a classical equivalence of abelian categories

$$\mathsf{Coh}\, \mathbb{P}^{r-1}_k \simeq \mathsf{Mgr}\, A \big/ \mathsf{Mfgr}\, A,$$

in which $\mathsf{Mgr}\, A$ stands for the category of positively graded A-modules and $\mathsf{Mfgr}\, A$ the Serre subcategory of finite length objects. By the localisation theorem [Sri96, 4.6], we obtain a long exact sequence relating their K-groups.

The projection morphism $A \to k$ permits us to consider graded k-vector spaces as graded A-modules. By the dévissage theorem [Sri96, 4.8], this inclusion induces an isomorphism of K-groups

$$K_i(\mathsf{Mgr}\, k) \simeq K_i(\mathsf{Mfgr}\, A).$$

Moreover we saw last time [Sri96, 5.4] that we had

$$K_i(\mathsf{Mgr}\, k) \simeq G_i(k) \otimes_{\mathbb{Z}} \mathbb{Z}[T] \quad \text{as well as} \quad K_i(\mathsf{Mgr}\, A) \simeq G_i(k) \otimes_{\mathbb{Z}} \mathbb{Z}[T],$$

in which T acts in both side by shifting $[-1]$. Altogether, we end up with the following diagram summarizing the situation:

$$\begin{tikzcd}
\dots \arrow[rr] & & K_i(\mathsf{Mfgr}\, A) \arrow[r] & K_i(\mathsf{Mgr}\, A) \arrow[rr] & & G_i(\mathbb{P}^{r-1}_k) \arrow[r] & \dots \\
& w \otimes_k k[-n] & & & w \otimes_k A[-n] & & \\
& & K_i(\mathsf{Mgr}\, k) \arrow[uu, "\cong"] & & & & \\
& w \otimes T^n \arrow[uu, mapsto] & & & w \otimes T^n \arrow[uu, mapsto] & & \\
& & G_i(k) \otimes \mathbb{Z}[T] \arrow[uu, "\cong"] \arrow[r, dashed, "h"] & G_i(k) \otimes \mathbb{Z}[T]. \arrow[uuuu, "\cong"'] & & &
\end{tikzcd}$$

It remains to understand this mysterious map h that we obtain by composing the rest of the morphisms. We will show that it is injective and has the expected cokernel by finding an explicit expression for it.

A first observation is that every map in the square commutes with shift, thus h is a morphism of $\mathbb{Z}[T]$-modules and it suffices to understand $h(w)$ for a class w in $G_i(k)$. In fact, there exists a graded free resolution (the ***Koszul resolution*** of k for A) of the form

$$0 \to A[-r] \otimes_k \bigwedge^r \mathcal{E} \to A[1-r] \otimes_k \bigwedge^{r-1} \mathcal{E} \to \dots \to A[-1] \otimes_k \mathcal{E} \to A \to k \to 0,$$

which by tensoring over k induces an exact sequence of functors $\mathsf{M}(k) \to \mathsf{Mgr}\, A$. We saw previously [Sri96, 4.5] that upon passing to K-groups this becomes an alternating sum of morphisms. Thus for any w in $G_i(k)$, we have

$$h(w) = w \cdot \sum_{i=0}^{r} (-1)^i \cdot T^i \cdot \left(\dim_k \bigwedge^i \mathcal{E} \right) = w \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} T^i.$$

In particular h is injective, so that

$$G_i(\mathbb{P}^{r-1}_k) \simeq \operatorname{coker} h \simeq G_i(k)^r = K_0(\mathbb{P}^{r-1}_k) \otimes_{K_0(k)} G_i(k)$$

with the composite isomorphism of the form announced in theorem statement. $\square$

4 The filtration by codimension of support

Recall the following definitions. For a closed subset Z of a Noetherian scheme X, we define the ***codimension*** of Z in X to be

$$\operatorname{codim}(Z, X) = \inf_{z:Z} \dim \mathscr{O}_{X,z}.$$

Given a coherent sheaf $\mathscr{F}$, its ***support*** is the closed subscheme $\operatorname{Supp} \mathscr{F}$ defined by the ideal sheaf $\operatorname{Ann} \mathscr{F}$. We write $\mathsf{Coh}^p(X)$ for the (Serre!) subcategory of coherent sheaves whose support is of codimension at least p. A few properties following from the definitions, see [Sri96, p. 64] for details:

- We have $\mathsf{Coh}^p(X) = \varinjlim_{\operatorname{codim} Z \geq p} \mathsf{Coh}\, Z$, and in particular $K_i(\mathsf{Coh}^p) = \varinjlim_{\operatorname{codim} Z \geq p} G_i(Z)$ for any $i \geq 0$.
- If $f : Y \to X$ is a flat map of Noetherian schemes, and $Z \subseteq X$ has codimension at least p, then $f^{-1}Z \subseteq Y$ has codimension at least p (by going down). In particular $f^* \mathsf{Coh}^p X \subseteq \mathsf{Coh}^p Y$.
- Let $(X_i)_I$ is a cofiltered system of Noetherian schemes with affine flat transition morphisms, and $X = \varprojlim_I X$ be Noetherian. Then for any $q \geq 0$ we have $K_q(\mathsf{Coh}^p X) = \varinjlim_I K_q(\mathsf{Coh}^p X_i)$.

Definition 16. *Let X a Noetherian scheme. The **filtration by codimension of support** is the decreasing filtration on $G_i(X)$ defined by*

$$F^p G_i(X) = \operatorname{im}\left[K_i(\mathsf{Coh}^p X) \to G_i(X)\right].$$

Note that this is a finite filtration whenever X has finite dimension.

The rest of chapter 5 depends heavily on spectral sequences, so we shall stop here!

Next next time, in K-theory working group:

Theorem 17. *Let $X^{(p)} \subseteq X$ the set of point of codimension p in X. There is a spectral sequence (of cohomological type)*

$$E_1^{p,q}(X) = \coprod_{x:X^{(p)}} K_{-p-q}(k(x)) \Rightarrow G_{-p-q}(X),$$

such that the induced filtration of $G_n(X)$ is the filtration by codimension of support. The spectral sequence is contravariant for flat morphisms. Further, if $(X_i)_I$ is a cofiltered system of Noetherian schemes with affine flat transition maps and $X = \varprojlim X_i$ is Noetherian, then the spectral sequence for X is the colimit of the spectral sequences for X_i.

References

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