Spectral Sequences
in Algebraic Topology

Notes for the Algebraic K-Theory Lecture Group

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Ale Fenu

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Contents

Chapter 1

Spectral sequences

Let R be a ring.
Classically, we define (C_*, d) as a graded chain complex over R and (C^*, d) as a graded cochain complex over R. We define $H_*(C_*) = \ker d_i/\operatorname{Im} d_{i+1}$ and $H^*(C^*) = \ker d^i/\operatorname{Im} d^{i-1}$. We will use this notation henceforth.

Definizione 1.0.1. A *filtration* on an R-module A is a family of submodules defined in one of the following ways:

- $\dots \subset F^{p+1}A \subset F^pA \subset \dots$ called a decreasing family;
- $\dots \subset F_{p-1}A \subset F_pA \subset \dots$ called an increasing family.

Definizione 1.0.2. A complex (C_*, d) is called a *filtered complex* if it is a filtered R-module, $F_iC_* = \bigoplus_j F_iC_j$ and $d(F_iC_*) \subset F_iC_*$.

Esempio 1.0.1. For $A = \mathbb{Z}$, we can define $F^iA = 2^i\mathbb{Z}$ for every $i \geqslant 0$ and $\mathbb{Z}$ for $i < 0$.

Definizione 1.0.3. Given a filtered R-module (A, F), we define its *associated graded module* as

$$E_0^p(A) = F^pA/F^{p+1}A \qquad \text{if } F \text{ is decreasing;}$$

$$E_p^0(A) = F_pA/F_{p-1}A \qquad \text{if } F \text{ is increasing.}$$

We observe that in general, even if F is a bounded filtration, i.e., eventually 0 and A in the right direction, the module $E_p^0(A)$ does not determine A (*extension problem*).

Esempio 1.0.2. Given a CW-complex X and its skeleton $X^{(p)}$, we can endow the complex of its singular (or cellular) chains $C_*(X)$ with a filtered module structure by defining

$$F_pC_* = C_*(X^{(p)}) \qquad \text{[increasing filtration]}$$

and for $C^* = Hom(C_*(X), R)$ we consider

$$F^pC^* = \{\phi \in C^* \,|\, F_{p-1}C_* \subset \ker \phi\} = Ann(F_{p-1}C_* \subset C_*) \qquad \text{[decreasing filtration]}$$

For the two cases, we obtain

$$E_p^0(C_*) = F_pC_*/F_{p-1}C_* = C_*(X^{(p)}, X^{(p-1)})$$

and

$$E_0^p(C_*) = Ann(F_{p-1}C_*)/Ann(F_pC_*) = C^*(X^{(p)}, X^{(p-1)}).$$

Esempio 1.0.3. Let $f : D_* \to C_*$ be a map of chain complexes and let F_p be an increasing filtration on C_*. Then it is possible to define an increasing filtration G_p on D_* as $G_pD_* :\ = f^{-1}(F_pC_*)$.

We observe that if (C_*, d) is a chain complex with an increasing filtration F, then $H_*(C_*)$ is also a filtered module:

$$F_pH_*(C_*) :\ = \mathrm{Im}(H_*(F_pC_*) \to H_*(C_*)) \quad \text{[increasing filtration]}$$

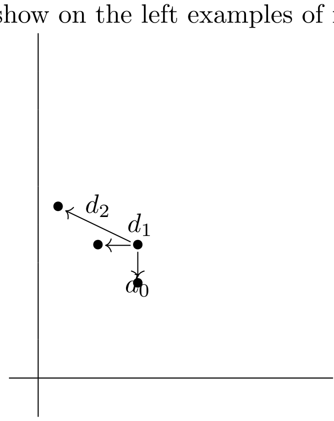
while if (C^*, d) is a filtered cochain complex, then $H^*(C^*)$ is also one by

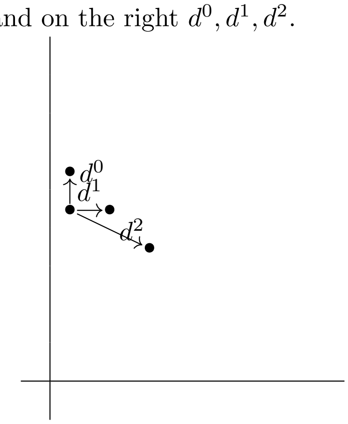
$$F^pH^*(C^*) :\ = \ker\{H^*(C^*) \to H^*(F^{p-1}C^*)\}$$

and this filtration is decreasing.

Definizione 1.0.4. An R-module is called *bigraded* if it is a sum of modules with degrees defined by a pair of integers (s, t). Such a module is called *differential* if it has maps $d : E \to E$ such that $d^2 = 0$ of bidegree $(-r, r-1)$ or $(r, 1-r)$ with cohomological indices.

We show on the left examples of maps d_0, d_1, d_2 and on the right d^0, d^1, d^2.





If we define

$$K^i :\ = \left(\bigoplus_{s+t=i} E^{s,t}, d\right)$$

we obtain a cochain complex (and similarly for chains).
We set

$$H_{p,q}(E_{*,*}, d) :\ = \ker\{d : E_{p,q} \to E_{p-r,q+r-1}\}/\mathrm{Im}\{d : E_{p+r,q-r+1} \to E_{p,q}\}$$

and similarly for cohomology.

Definizione 1.0.5. A *spectral sequence* is a collection of bigraded differential R-modules $\{E^r_{p,q}, d^r\}_r$ with $r \geqslant k$ with d^r of bidegree $(r, 1-r)$ such that

$$E^{r+1}_{p,q} = H_{p,q}(E^r, d^r).$$

An important observation is that (E_r, d_r) determines E_{r+1} but not d_{r+1} (we omit the bidegree).

Definizione 1.0.6 (Limit of a spectral sequence)**.** Setting $E_{i+1} = Z_i/B_i$ and $d_{i+1} : Z_i/B_i \to Z_i/B_i$ we can identify Z_{i+1} as a quotient of a submodule of Z_i and similarly for B_i (with reversed order).
We can set $B_\infty : = \bigcup B_i, Z_\infty = \bigcap Z_i$ and define as the *limit of the spectral sequence*

$$E_\infty = Z_\infty / B_\infty.$$

Definizione 1.0.7 (Convergence of a spectral sequence)**.** A spectral sequence $(E_r^{p,q}, d_r)$ is said to *converge* to H^* if there exists a decreasing filtration F on H^* such that $E_\infty^{p,q} \cong E_0^p(H^{p+q})$.

Under suitable hypotheses (for example $E_2^{p,q} \neq 0$ only for $p, q \geqslant 0$) the differentials d_r are eventually zero for fixed p, q and therefore a spectral sequence converges in a stronger sense: we will say that it *stabilizes*.

Definizione 1.0.8. We say that (E_r, d_r) **collapses** at page E_N if $d_r \equiv 0 \, \forall r \geqslant N$.

Let F be an increasing filtration on a graded R-module A.

Definizione 1.0.9. We will say that F is a *convergent filtration* if

$$\bigcup_j F_j A = A, \bigcap_s F_s A = 0.$$

Definizione 1.0.10. We will say that F is a filtration

- bounded below if $\forall\ t\ \exists\ s(t)$ such that $F_{s(t)} A_t = 0$;
- bounded above if $\forall\ t\ \exists\ s'(t)$ such that $F_{s'(t)} A_t = A_t$.

Teorema 1.0.4. *A graded filtered chain complex (A, d, F) determines a spectral sequence $\{E^r_{s,t}, d^r\}_{r \geqslant 1}$ such that $E^1_{s,t} = H_{s+t}(F_s A / F_{s-1} A)$ and where d^1 is the boundary operator (connecting homomorphism) of the triple $(F_s A, F_{s-1} A, F_{s-2} A)$.*
If F is convergent and bounded below and above then the spectral sequence stabilizes.
The limit $E^\infty_{p,q}$ is isomorphic to $F_p H_{p+q}(A)/F_{p-1} H_{p+q}(A)$.

An analogous statement holds for cochain complexes, determining a cohomology spectral sequence.

Proof. We omit the homological degree.
We set

$$Z^r_s = \{c \in F_s A \,|\, dc \in F_{s-r} A\}$$

$$Z^\infty_s = \{c \in F_s A \,|\, dc = 0\}$$

$$E^r_s = Z^r_s/(Z^{r-1}_{s-1} + dZ^{r-1}_{s+r-1})$$

$$E^\infty_s = Z^\infty_s/(Z^\infty_{s-1} + dA \cap F_sA)$$

and we observe that $d : Z^r_s \to Z^r_{s-r}$ of bidegree $(-r, r-1)$ induces d^r.
In particular

$$E^0_s = F_sA/F_{s-1}A$$

$$d^0 : F_sA/F_{s-1}A \to F_sA/F_{s-1}A \quad \text{ is induced by } d \text{ on the quotient}$$

$$E^1_s = Z^1_s/(Z^0_{s-1} + dZ^0_s)$$

furthermore Z^1_s/Z^0_s are cycles of $F_sA/F_{s-1}A$ while $(Z^0_{s-1} + dZ^0_s)/Z^0_{s-1}$ are boundaries of $F_sA/F_{s-1}A$. Putting everything together

$$E^1_{s,t} \cong H_{s+t}(F_sA/F_{s-1}A)$$

and the isomorphism is induced by $Z^1_{s,t} \to F_sA$. The differential is instead induced by the boundary map.

$$\begin{tikzcd}
 & & 0 \\
0 \arrow[d] & F_sA_{s+t}/F_{s-2}A_{s+t} \arrow[r] \arrow[d, "d"] & F_sA_{s+t}/F_{s-1}A_{s+t} \arrow[u] \\
F_{s-1}A_{s+t-1}/F_{s-2}A_{s+t-1} \arrow[r] & F_sA_{s+t-1}/F_{s-2}A_{s+t-1} & \\
 & & Z^1_{s,t} \arrow[uu] \arrow[uul, bend right] \arrow[dl, "d"] \\
 & Z^1_{s-1,t} \arrow[uu] \arrow[uul] &
\end{tikzcd}$$

We now prove that $H_*(E^r) = E^{r+1}$.

$$\begin{aligned} &\ker(d^r : E^r_s \to E^r_{s-r}) = \\ &= \{c \in Z^r_s | dc \in Z^{r-1}_{s-r-1} + dZ^{r-1}_{s-1}\}/(Z^{r-1}_{s-1} + dZ^{r-1}_{s+r-1}) = \\ &= (Z^{r+1}_s + Z^{r-1}_{s-1})/(\text{same denominator}). \end{aligned}$$

Furthermore

$$\operatorname{Im}(d^r : E^r_{s+r} \to E^r_s) = (dZ^r_{s+r} + Z^{r-1}_{s-1})/(Z^{r-1}_{s-1} + dZ^{r-1}_{s+r-1}).$$

We obtain

$$\ker d^r/\operatorname{Im} d^r = (Z^{r+1}_s + Z^{r-1}_{s-1})/(dZ^r_{s+r} + Z^{r-1}_{s-1}) = E^{r+1}_s$$

which is what we wanted.
It remains to investigate the limit of the spectral sequence.

$$E^r_s = Z^r_s/(dZ^{r-1}_{s+r-1} + Z^{r-1}_{s-1}) \cong (Z^r_s + F_{s-1}A)/(F_{s-1}A + dZ^{r-1}_{s+r-1}).$$

Calculating the limit

$$E_s^\infty = \bigcap_r (Z_s^r + F_{s-1}A) / \bigcup_r \left(F_{s-1}A + dZ_{s+r-1}^{r-1}\right) = Z_s^\infty / \left(Z_{s-1}^\infty + dA \cap F_s A\right).$$

Since the filtration is bounded, we have that $\forall s, t\, \exists r : E_{s,t}^\infty = E_{s,t}^{r(s,t)}$.
$F_s H_{s+t}(A) = \mathrm{Im}[H_{s+t}(F_s A) \to H_{s+t}(A)]$ and therefore $F_s H_*(A) = Z_s^\infty/(dA \cap F_s A)$ and furthermore

$$F_s H_*(A)/F_{s-1}H_*(A) = E_s^\infty.$$

□

1.1 Examples of applications of spectral sequences

Proposizione 1.1.1. *Let X be a CW complex, then $H_*^{cell}(X) = H_*(X)$, i.e., cellular and singular homology coincide.*

Proof. Let $C_*(X)$ be the complex of singular chains on X. Consider the filtration $F_p C_*(X) = C_*(X^{(p)})$.
Let us look at $E_{p,q}^0 = C_{p+q}(X^{(p)})/C_{p+q}(X^{(p-1)})$ and therefore $E_{p,q}^1 = H_{p+q}(X^{(p)}, X^{(p-1)})$ which is equal to $C_p^{cell}(X)$ if $q = 0$ and zero otherwise.
By definition of cellular chains we have that the boundary map $\partial : C_p^{cell}(X) \to C_{p-1}^{cell}(X)$ is the boundary map of the triple $(X^{(p)}, X^{(p-1)}, X^{(p-2)})$ coming from the snake lemma $\partial : H_p(X^{(p)}, X^{(p-1)}) \to H_{p-1}(X^{(p-1)}, X^{(p-2)})$ and coincides with the differential d^1 of $E_{p,q}^1$.
It follows that $E_{p,q}^2 = H_p^{cell}(X)$ for $q = 0$ and zero otherwise.
It holds that $E^2 = E^\infty$ which is equal to $H_*(X)$ by the previous theorem. □

Proposizione 1.1.2. *Let $(C_*, d), (C'_*, d')$ be chain complexes over the field $\mathbb{K}$. It holds that*

$$H_d(C \otimes C') = [H_*(C_*) \otimes H_*(C')]_d.$$

Proof. Consider the complex $D_* = C_* \otimes C'_*$ with $D_p = \bigoplus_{i+j=p} C_i \otimes C'_j$ and on it the filtration

$$F_p(C \otimes C')_q = \bigoplus_{i \leqslant p} C_i \otimes C'_{q-i}.$$

Then the following hold

$$E_{p,q}^0 = C_p \otimes C'_q$$

$$d^0 = (-1)^p Id_C \otimes d'$$

$$E_{p,q}^1 = C_p \otimes H_q(C')$$

$$d^1 = d \otimes Id$$

$$E_{p,q}^2 = H_p(C) \otimes H_q(C')$$

$$d^2 = 0 = d^n \text{ for } n > 2.$$

The spectral sequence then stabilizes at E^2 and $H_d(C \otimes C') = \bigoplus_{p+q=d} H_p(C) \otimes H_q(C')$ by

the theorem.
This is the thesis. $\square$

1.2 Cech-de Rham Isomorphism

Let X be a topological space and $\mathcal{U}$ a cover of it.

Definizione 1.2.1. For $U_{i_0}, U_{i_1}, \ldots, U_{i_k} \in \mathcal{U}$ we denote by $U_{i_0,i_1,\ldots,i_k}$ their intersection and define the *Cech complex* for $(X,\mathcal{U})$:

- $\check{C}^k_{\mathcal{U}}(X) = \prod_{i_0 \leqslant \cdots \leqslant i_k} \overbrace{C(U_{i_0,\ldots,i_k}, \mathbb{R})}^{\text{locally constant functions}}$;
- $\partial^k : \check{C}^k_{\mathcal{U}}(X) \to \check{C}^{k+1}_{\mathcal{U}}$ defined by

$$(\partial^k \alpha)_{i_0,\ldots,i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \alpha_{i_0,\ldots,\hat{i_j},\ldots,i_{k+1}}$$

Clearly $\partial^2 = 0$ and consequently we define

Definizione 1.2.2. $\check{H}^*(X,\mathcal{U}) := H^*(\check{C}^*_{\mathcal{U}}, \partial)$ is the *Cech cohomology.*

Now we introduce the de Rham complex.

Definizione 1.2.3. For M being a C^∞ n-manifold we denote by $\Omega^k(M)$ the space of C^∞ differential k-forms on M. Locally $f \in \Omega^k(M)$ is written as

$$f = \sum_{|I|=k} f_I(x) dx^I$$

with $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. The boundary map acts locally as

$$df = \sum_{j=1}^{n} \frac{\partial}{\partial x^j} f(x)\ dx^j \wedge dx^I$$

and satisfies $d^2 \cong 0$. We then call $(\Omega^*(M), d)$ the *de Rham complex* associated to M and $H^*_{dR}(M) := H^*(\Omega^*(M), d)$ *de Rham cohomology* of M.

Proposizione 1.2.1 (Poincaré Lemma)**.** *If $A \subset \mathbb{R}^n$ is a contractible open set then $H^0_{dR}(A) = \mathbb{R}$ and $H^i_{dR}(A) = 0$ for $i > 0$, i.e., every closed form is exact.*

Definizione 1.2.4. A cover $\mathcal{U}$ is called a *good cover* if all finite intersections of open sets of $\mathcal{U}$ are contractible.

Proposizione 1.2.2. *Let $\mathcal{U}$ be a good cover of M (paracompact C^∞ n-manifold). Then*

$$\check{H}^*(M,\mathcal{U}) \cong H^*_{dR}(M).$$

Proof. Consider the Cech-de Rham double complex

$$C^{p,q} := \prod_{i_0 \leqslant \cdots \leqslant i_p} \Omega^q(U_{i_0,\ldots,i_p}) \qquad\qquad D = \partial + (-1)^p d$$

and consider two different filtrations of (C^*, d) with $C^n = \bigoplus_{p+q=n} C^{p,q}$:

- $F^p C^n = \bigoplus_{i \geqslant p} C^{i,n-i}$. We observe the spectral sequence $E_0^{p,q} = C^{p,q}$ and $d_0 = (-1)^p d$. We derive $E_1^{p,q} = \prod H^q(U_{i_0,\ldots,i_p})$ which is trivial for $q > 0$ since the cover consists of open sets with contractible finite intersections. In particular, only the terms $H^0(U_{i_0,\ldots,i_p}) = C(U_{i_0,\ldots,i_p}, \mathbb{R})$ survive.
  The sequence stabilizes at $E_2 = E_\infty$ from which $H^*(C^*, d) \cong \check{H}^*(M, \mathcal{U})$ follows.

- if instead we filter with

  $$F^p C^n = \bigoplus_{i \geqslant p} C^{n-i,i}$$

  and define $E_0^{p,q} = C^{p,q}$, $d_0 = \partial$ (the Cech differential of $\prod \Omega^q(U_{i_0,\ldots,i_p})$) we can use the following fact (proved subsequently):

  Proposizione 1.2.3. *There is a chain homotopy $k : \prod \Omega^*(U_{i_0,\ldots,i_p}) \to \prod \Omega^*(U_{i_0,\ldots,i_{p-1}})$ between the identity and the zero map in degree $p > 0$.*

  It follows that in the spectral sequence, at page 1 only $E_1^{p,q}$ for $p = 0$ survives, given by $\ker \partial$ (global forms).

  $$\begin{array}{c|cc}
  & \cdots & \\
  & \Omega^2(M) & \cdots \\
  & \Omega^1(M) & \cdots \\
  & \Omega^0(M) & \cdots \\
  \hline
  & 0 & \cdots
  \end{array}$$

  $$\Rightarrow E_2^{p,q} = H^q_{dR}(M)$$

  if p=0, zero otherwise. The spectral sequence therefore stabilizes at $E_2 = E_\infty$ and consequently $H^*(C^*, d) \cong H^*_{dR}(M)$.

$\square$

We prove **Proposition 1.2.3**.

Proof. Let $\{\rho_i\}$ be a partition of unity associated to $\mathcal{U}$. Setting

$$(k\omega)_{i_0,\ldots,i_{p-1}} := \sum_i \rho_i \omega_{i,i_0,\ldots,i_{p-1}}$$

one verifies that

$$(\partial k\omega)_{i_0,\ldots,i_p} = \sum (-1)^j \rho_i \omega_{i,i_0,\ldots,\hat{i_j},\ldots,i_p}$$

and

$$(k\partial\omega)_{i_0,\ldots,i_p} = \omega_{i_0,\ldots,i_p} + \sum (-1)^{j+1} \rho_i \omega_{i,i_0,\ldots,\hat{i_j},\ldots,i_p}$$

therefore $\partial k + k\partial = Id$ and thus k is a chain homotopy as desired. $\square$

1.3 Pairs and spectral sequences

Consider the filtration $\varnothing \subset X_0 \subset X_1$ associated to the pair (X,Y) with $X = X_1$ and $X_0 = Y$. This is a filtration with only two terms and analogously to what has already been done we obtain:

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) = \begin{cases} H_q(Y) \text{ if } p = 0 \\ H_{q+1}(X,Y) \text{ if } p = 1. \end{cases}$$

We observe page 1.

$$H_q(Y) \xleftarrow{d^1} H_{q+1}(X,Y)$$

0 1

Where $d^1 : E^1_{1,q} \to E^1_{0,q}$ is the connecting homomorphism of the long exact sequence of the pair (X,Y).

We want to compare the term E_∞ and E_2 (which we know to be equal since the spectral sequence stabilizes immediately).
Let us write them both out explicitly.

$$E^\infty_{0,q} : \frac{\operatorname{Im}(H_q(X_0) \to H_q(X_1))}{\operatorname{Im}(H_q(\varnothing) \to H_q(X_1))} = \operatorname{Im}(i : H_q(Y) \to H_q(X)),$$

$$E^\infty_{1,q} = \frac{H_{q+1}(X_1)}{\operatorname{Im}(H_{q+1}(X_0) \to H_{q+1}(X_1))} = \frac{H_{q+1}(X)}{\operatorname{Im}(i : H_{q+1}(Y) \to H_{q+1}(X))}.$$

For E^2 the calculations give

$$E^2 = \ker(d^1 : H_{q+1}(X,Y) \to H_q(Y)) \quad (p = 1),$$

$$E^2 = H_q(Y)/\operatorname{Im} d^1 \quad (p = 0).$$

Putting everything together we obtain the exactness of the long exact sequence of the pair in homology.

Chapter 2

Serre Sequence

Let $E \xrightarrow{\pi} B$ be a Serre fibration, which we recall means solving the lifting problem

$$\begin{tikzcd} I^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ I^{n+1} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

From now on we will suppose B is a path-connected CW complex, a hypothesis that is no more restrictive than B being path-connected provided we use cellular approximation and pullback of bundles.

Definizione 2.0.1. A *local system* $\mathcal{G} = \{G_x, \tau_\gamma\}$ of groups on a topological space X is a functor that assigns to each $x \in X$ a group C_x and to each path $\gamma : I \to X$ from x_0 to x_1 a morphism $\tau_\gamma : G_{x_0} \to G_{x_1}$ which depends only on the homotopy class with fixed endpoints of γ and such that if γ is constant then τ_γ is the identity. In other words, a local system is a functor from the fundamental groupoid to groups.

We observe that clearly τ_γ is an isomorphism and thus for X path-connected all G_x are isomorphic.
The $\pi_1(X, x_0)$ acts on the left on G_{x_0}.

Definizione 2.0.2. If X is path-connected and its fundamental group acts trivially on $\mathcal{G}$, we will say that the local system is trivial.

Let us assume that X is path-connected and admits a universal cover $\widetilde{X}$. We know that $\pi_1(X, x_0)$ acts on the right by translation on $\widetilde{X}$ (monodromy action) and consequently acts (again from the right) on the complex $C_*(\widetilde{X})$ and this action commutes with the differential.

Definizione 2.0.3. Setting $G = G_{x_0}$ abelian, we can define the complexes

$$C_*(X, \mathcal{G}) \colon = C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} G$$

$$C^*(X,\mathcal{G})\colon = Hom_{\mathbb{Z}[\pi_1(X,x_0)]}(C_*(\widetilde{X},G))$$

which we call singular chains and cochains with coefficients in the local system $\mathcal{G}$.

Definizione 2.0.4. We define homology and cohomology with local coefficients as the homology of the first or second complex.

We now give an important theorem.

Teorema 2.0.1 (Serre Spectral Sequence)**.** *Let $\pi : E \xrightarrow{\pi} B$ be a Serre fibration, there exists a first-quadrant spectral sequence $E^r_{p,q}$ with $r \geqslant 2$ with*

$$E^2_{p,q} = H_p(B;\{H_q(E_x;M)\}) = H_p(B;\mathcal{H}_q(\pi^{-1};M))$$

converging to $E^\infty_{p,q} \Rightarrow H_{p+q}(E;M)$ for a suitable filtration of $H_*(E)$.

Esempio 2.0.2. $SU(3)$
Let us consider the fibration $SU(2) \hookrightarrow SU(3) \to S^5$ given by the action of $SU(3)$ on the 5-sphere. Here too $E^2 = E^\infty$.

$$\begin{array}{c|cc}
3 & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} \\
\hline
 & 0 & 5
\end{array}$$

We obtain

$$H_p(SU(3);\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0,3,5,8 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, on every diagonal $p+q=n$ of the E^∞ page only a single non-zero term appears and thus the associated graded module of the homology of the total space coincides with the homology. This reasoning will be used often.

Esempio 2.0.3. $SU(4)$
Here too, we consider the fibration $SU(3) \hookrightarrow SU(4) \to S^7$. In a completely analogous manner to before, we derive $E^2 = E^\infty$, from which

$$\begin{array}{c|cc}
8 & \mathbb{Z} & \mathbb{Z} \\
5 & \mathbb{Z} & \mathbb{Z} \\
3 & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} \\
\hline
 & 0 & 7
\end{array}$$

$$H_p(SU(4);\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0,3,5,7,8,10,12,15 \\ 0 & \text{otherwise} \end{cases}$$

We have again used the fact that in the E^∞ page only a single non-zero term appears on every diagonal $p+q=n$, therefore it coincides with the homology.

Esempio 2.0.4. $SU(5)$ from the fibration

$$SU(4) \hookrightarrow SU(5) \to S^9$$

$$
\begin{array}{c|cc}
15 & \mathbb{Z} & \mathbb{Z} \\
12 & \mathbb{Z} & \mathbb{Z} \\
10 & \mathbb{Z} & \mathbb{Z} \\
9 & \mathbb{Z} & \mathbb{Z} \\
7 & \mathbb{Z} & \mathbb{Z} \\
3 & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} \\
\hline
 & 0 & 9
\end{array}
$$
.

We observe that $E^2 = \cdots = E^9$, while $E^{10} = E^\infty$ but the previously calculated method is not sufficient to understand to what these pages are isomorphic. Indeed, observing the diagonal $p + q = 12$ we notice that in the infinity page we do not know which extension problem we have to solve, having two non-zero terms.

Now let us observe other examples based on the path fibration (the *pathspace fibration*).

Esempio 2.0.5. ΩX. Let us take a pointed, path-connected and simply connected topological space (X, x_0) and consider the fibration $\Omega X \to PX \to X$. We observe that PX is contractible.

$$E^2_{p,q} = H_p(X, H_q(\Omega X)) \Rightarrow E^\infty_{p,q} = \begin{cases} \mathbb{Z} & p = q = 0 \\ 0 & \text{otherwise since } PX \text{ is contractible.} \end{cases}$$

$$
\begin{array}{c|ccccccccc}
2n-2 & & & & & & & & & \\
r-1 & & & & & & & & & \\
n-1 & & & & \bullet & & & & & \\
0 & \mathbb{Z} & \dots & 0 & \bullet & \dots & \bullet & \dots & \bullet & \\
\hline
 & 0 & & & n & & r & & 2n &
\end{array}
$$

Therefore $d^r_{r,0} : \underbrace{E^r_{r,0}}_{=H_r(X)} \to \underbrace{E^r_{0,r-1}}_{=H_{r-1}(\Omega X)}$ must be an isomorphism for $r \leqslant 2n - 2$. To see this, it suffices to observe that all rows from 1 to $n - 2$ are also zero: the homology of ΩX in the first column is zero up to the $(n-2)$-th, otherwise it would survive until the E^∞ page,

In fact, a direct consequence of the example is the following proposition.

Proposizione 2.0.6. *If X is $(n-1)$-connected, i.e., it has vanishing homotopy from the zeroth to the $(n-1)$-th, then $H_1(X) = \cdots = H_{n-1}(X) = 0$ and furthermore $H_r(X) \cong H_{r-1}(\Omega X)$ for $r \leqslant 2n - 2$.*

A consequence is the following.

Teorema 2.0.7 (Hurewicz Theorem)**.** *If X is $(n-1)$-connected and $n \geqslant 2$ then $\pi_n(X) = H_n(X)$.*

As a final example we have

Esempio 2.0.8. ΩS^n
Since $\pi_i(S^n) = 0$ for $i < n$ and using $\Omega S^n \to PS^n \to S^n$ we obtain

$$
\begin{array}{c|cc}
\dots & \dots & \dots \\
3n-3 & \mathbb{Z} & \mathbb{Z} \\
2n-2 & \mathbb{Z} & \mathbb{Z} \\
n-1 & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} \\
\hline
 & 0 & n
\end{array}
$$

e dunque

$$H_i(\Omega S^n) = \begin{cases} \mathbb{Z} & (n-1)|i \\ 0 & \text{altrimenti.} \end{cases}$$

2.1 Comparison of spectral sequences

The association of a spectral sequence to a filtered graded chain complex is functorial, therefore maps of filtered graded complexes induce maps of spectral sequences.

Teorema 2.1.1. *Let $\tau : C \to C'$ be a map of filtered graded chain complexes* [1]*, with a convergent and bounded below filtration.*
If for some $r \geqslant 1$ we have that $\tau^r : E^r \to E'^r$ is an isomorphism then τ induces an isomorphism in homology

$$\tau_* : H_*(C) \xrightarrow{\approx} H_*(C').$$

Proof. By functoriality it follows that τ^l is an iso for $l \geqslant r$ and in particular $\tau^\infty : E^\infty \to E'^\infty$ is also one.
Now we can observe the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & F_{s-1}(H_n(C)) \arrow[r] \arrow[d] & F_s(H_n(C)) \arrow[r] \arrow[d] & E^\infty_{s,n-s} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{s-1}(H_n(C')) \arrow[r] & F_s(H_n(C')) \arrow[r] & E^\infty_{s,n-s} \arrow[r] & 0
\end{tikzcd}
$$

For fixed n we can take $\bar{s}$ sufficiently small so that the boundedness from below of the filtration guarantees $F_{\bar{s}-1}H_n(C) = F_{\bar{s}-1}H_n(C') = 0$ and by induction (using the diagram), for $s > \bar{s}$ we obtain

$$\tau_* : F_sH_n(C) \to F_sH_n(C')$$

isomorphism. Since the filtration F is convergent the result follows on the increasing union

$$\to \tau_* : \bigcup_s F_sH_n(C) = H_n(C) \to H_n(C') = \bigcup_s F_sH_n(C') \quad \text{is an isomorphism.}$$

□

[1] We observe that this hypothesis implies compatibility of the filtrations with respect to τ and is stronger than a simple morphism of complexes between graded and filtered complexes.

2.2 First proof of the Serre Spectral Sequence Theorem

Up to using the CW approximation theorem, the pullback of bundles and Whitehead's theorem we can suppose that B is a CW complex.
Let $B^{(i)}$ be the i-th skeleton of B, this induces the filtration

$$\varnothing \subset B^{(0)} \subset \cdots \subset B^{(n)} \subset \dots$$

and the pullback one on E given by $E^{(i)} = \pi^{-1}(B^{(i)})$. This topological filtration induces a filtration on the singular chain complexes $C_*(E)$ from which we obtain a spectral sequence with

$$E^0_{p,q} = C_{p+q}(E^p, E^{p-1})$$

and differential d^0 induced by the boundary of $C_*(E)$.

$$\begin{aligned}
\Rightarrow E^1_{p,q} &= H_{p+q}(E^p, E^{p-1}) = H_{p+q}(\pi^{-1}(B^{(p)}), \pi^{-1}(B^{(p-1)})) \overset{\text{excision}}{=} \\
&= H_{p+q}\left(\pi^{-1}(B^{(p)}), \pi^{-1}(B^{(p)}\setminus \bigcup_i \overbrace{\{c_i\}}^{\text{center of } p\text{-cell}})\right) = \\
&= H_{p+q}\left(\bigcup_i \pi^{-1}(D^p_i), \bigcup_i \pi^{-1}(D^p_i \setminus \{c_i\})\right) = \\
&= \bigoplus_i H_{p+q}(\pi^{-1}(D^p_i), \pi^{-1}(D^p_i \setminus int(D_i))).
\end{aligned}$$

Up to CW approximation we can suppose that the fibration π is trivial on the disk D_i with fiber $\pi^{-1}(c_i) = F_i$. Therefore

$$H_{p+q}(\pi^{-1}(D^p_i), \pi^{-1}(\partial D^p_i)) = H_{p+q}(D^p_i \times F_i, S^{p-1} \times F_i) \overset{\text{Kunneth}}{=} H_q(F_i).$$

Therefore $E^1_{p,q} = \bigoplus_i H_q(F_i)$.
Now we have two different cases. If the local system $\{H_q(\pi^{-1})\}$ is trivial, $E^1_{p,q}$ are precisely the cellular chains $C^{cell}_p(B, H_q(F)) = C^{cell}_p(B) \otimes H_q(F)$ with the differential d^1 induced locally by the map of the triple $(B^{(p)}, B^{(p-1)}, B^{(p-2)})$.
Alternatively, we can pass to the universal cover of B and obtain

$$E^1_{p,q} = C^{cell}_p(\widetilde{B}) \otimes_{\pi_1(B)} H_q(F) = C^{cell}_p(B; H_q(\pi^{-1})).$$

In both cases $E^2_{p,q} = H_p(B; H_q(\pi^{-1}))$ as desired.

2.3 Second proof of the Serre Spectral Sequence Theorem

Given the fibration $\pi : E \to B$ we consider

$$Sin_{s,t}(\pi) = \{(f, \sigma) | f : \Delta^s \times \Delta^t \to E,\ \ \sigma : \Delta^s \to B, \pi \circ f = \sigma \circ pr_1\}$$

$$\begin{array}{ccc} \Delta^s \times \Delta^t & \xrightarrow{f} & E \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\ \Delta^s & \xrightarrow{\sigma} & B \end{array}$$

and we obtain a functor $Sin_{*,*} : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$.
Let us take $RSin_{*,*}(\pi)$ the free R-module generated by $Sin_{s,t}(\pi)$.
There are two differentials:

$$\begin{aligned} \partial'_{p,q} : RSin_{p,q}(\pi) &\to RSin_{p-1,q} \\ (f,\sigma) &\mapsto \sum_{i=0}^{p} (-1)^i (f \circ (\varepsilon_i^p \times id_{D_q}), \sigma \circ \varepsilon_i^p) \end{aligned}$$

$$\begin{aligned} \partial''_{p,q} : RSin_{p,q}(\pi) &\to RSin_{p,q-1} \\ (f,\sigma) &\mapsto \sum_{j=0}^{q} (-1)^j (f \circ (id_{\Delta_p} \times \varepsilon_j^q), \sigma). \end{aligned}$$

The differentials commute therefore we can consider $d = \partial' + (-1)^q \partial''$ and then two filtrations on $RSin_{*,*}$ (as in the case of the *Cech-de Rham complex*). The two spectral sequences will converge to the same thing and we will conclude the thesis.

- First filtration: $F_p(RSin_{*,*}(\pi))_* = \bigoplus_{\substack{s+t=n \\ t \leqslant p}} RSin_{s,t}(\pi)$ and $d_0 = \partial'$.

  Let $(f,\sigma) \in Sin_{s,t}(\pi)$, we consider $\hat{f}$ associated via adjunction

$$\begin{array}{ccc} \Delta^s \times \Delta^t & \xrightarrow{f} & E \\ \downarrow{\scriptstyle pr_1} & & {\scriptstyle \pi}\downarrow \\ \Delta^s & \xrightarrow{\sigma} & B \end{array} \qquad\qquad \begin{array}{ccc} \Delta^s & \overset{\hat{f}}{\dashrightarrow} & E^{\Delta^t} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{c} & B^{\Delta^t} \end{array}$$

  where c is the inclusion map on the constant map.
  The data of f and $\hat{f}$ are equivalent.
  Consider the pullback

$$\begin{array}{ccc} E_t^1 & \longrightarrow & E^{\Delta^k} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[c]{} & B^{\Delta^k} \end{array}$$

  and therefore $RSin_{s,t}(\pi) = C_s(E_t^1; R)$.

  Furthermore $E_t^1 \to E^{\Delta^t}$ is a homotopy equivalence because $B \to B^{\Delta^t}$ is and $E \xrightarrow{c} E^{\Delta^t}$ is as well, so $C_*(E_t^1) \to C_*(E)$ induces isomorphisms in homology and

$$E_{s,t}^1 = H_s(E) \ \ \forall t \geqslant 0.$$

  The differential $d^1 = \pm\partial''$ is an alternating sum of restriction maps to the faces of Δ^t, which therefore induce the identity in homology. We conclude that

$$E^2_{s,t} = \begin{cases} H_s(E) & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Second filtration: $F_p(RSin_{*,*}(\pi))_* = \bigoplus_{\substack{s+t=n \\ s \leqslant p}} RSin_{s,t}(\pi)$ and differential $d_0 = \partial''$.

  Given $(f, \sigma) \in Sin_{s,t}(\pi)$ we have

$$\begin{tikzcd}
\Delta^s \times \Delta^t \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr, "pr_1"] & \sigma^{-1}(E) \arrow[r] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\
& \Delta^s \arrow[r, "\sigma"] & B
\end{tikzcd}$$

so, after fixing $\sigma : \Delta^s \to B$ we have by adjunction

$$\begin{tikzcd}
\Delta^t \arrow[r, dashed] \arrow[rr, bend left, "f"] \arrow[dr, bend right] & \hat{\pi}^{-1}(*) \arrow[r] \arrow[d] & E^{\Delta^s} \arrow[d, "\hat{\pi}"'] \\
& \{*\} \arrow[r, hook, "j"] & B^{\Delta^s}
\end{tikzcd}
\qquad
\begin{array}{c} \hat{\pi}^{-1}(*) = \Gamma(\Delta^s, \sigma^{-1}(E)) \\[2em] j(*) = \sigma \end{array}$$

so

$$E^0_{s,t} = \bigoplus_{\sigma : \Delta^s \to B} C_t(\Gamma(\Delta^s, \sigma^{-1}(E))),$$

$$E^1_{s,t} = \bigoplus_{\sigma} H_t(\Gamma(\Delta^s, \sigma^{-1}(E)))$$

and d^1 is the alternating sum of the faces of the simplicial module $E^1_{s,t}$ which act in this way: if $\phi : [s'] \to [s]$ is a map in the simplicial category, it induces

$$\phi^* : \Gamma(\Delta^s, \sigma^{-1}(E)) \to \Gamma(\Delta^{s'}, (\sigma \circ \phi)^{-1}(E))$$

and therefore $\phi^* : E^1_{s,t} \to E^1_{s',t}$.
Since $\pi : E \to B$ is a fibration and Δ^s is contractible, $\sigma^{-1}(E) \to \Delta^s$ is a trivial fibration as it is the pullback of a fibration, over a contractible base.
We thus have a homotopy equivalence between

$$\Gamma(\Delta^s, \sigma^{-1}(E)) \cong F_\sigma^{\Delta^s} \cong F_\sigma$$

with F_σ being the fiber of π over the vertex 0 of σ.
Therefore

$$E^1_{s,t} = \bigoplus_{\sigma \in Sin_s(B)} H_t(F_\sigma).$$

If B is simply connected, we can fix isomorphisms between the various $H_t(F_\sigma)$ as σ varies, so

$$E^1_{s,t} = C_s(B) \otimes H_t(F).$$

Otherwise we can pass to the universal cover $\widetilde{B}$ of B and we have

$$E^1_{s,t} = C_s(\widetilde{B}) \otimes_{\pi_1(B)} H_t(F) = C_s(B; H_t(\pi^{-1}))$$

and d^1 is the boundary map of $C_*(\widetilde{B})$ and therefore we conclude in any case

$$E^2_{s,t} = H_s(B; H_t(\pi^{-1}))$$

which concludes the proof.

2.4 Serre Spectral Sequence in Cohomology

Teorema 2.4.1. *Let $F \to E \to B$ be a Serre fibration, let G be a group and B path-connected such that $\pi_1(B)$ acts trivially on $H^*(F; G)$.*
There exists a first-quadrant spectral sequence such that

$$E_2^{p,q} = H^p(B; H^q(F; G))]$$

$$E_\infty^{p,n-p} \cong F_p^n / F_{p+1}^n$$

for an appropriate filtration

$$0 \subset F_n^n \subset \cdots \subset F_0^n = H^n(E; G).$$

Proof. The proof can be carried out analogously to those done for the homological version of the theorem. □

2.5 Multiplicative properties of the Serre spectral sequence in cohomology

Let the coefficient group be a ring R.
We assume that $H^p(B, H^q(F; R)) \cong H^p(B; R) \otimes H^q(F; R)$ (for example if $H^*(B; R)$ or $H^*(F; R)$ are free or if R is a field).
Then E_2 has a multiplicative structure:

$$E_2 = H^*(B; R) \otimes H^*(F; R)$$

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b| \cdot |a'|} \cdot aa' \otimes bb'.$$

Furthermore, if A is a filtered ring with a decreasing filtration $F_i A$ such that $F_p A \cdot F_q A \subset F_{p+q} A \Rightarrow GA = \bigoplus (F_p A / F_{p+1} A)$ has a ring structure.
This structure can be quite poor, for example if $F_p A \cdot F_q A \subset F_{p+q+1} A$ then GA has a trivial product structure.
We recall that the multiplicative structure in $H^*(X)$ is given by the diagonal and Kunneth maps:

$$H^*(X) \otimes H^*(X) \to H^*(X \times X) \xrightarrow{\Delta} H^*(X)$$

and for relative chains

$$c_1 \in H^k(X, X_1) \quad c_2 \in H^l(Y, Y_1)$$

$$c_1 \otimes c_2 \in H^{k+l}(X \times Y, X_1 \times Y \cup X \times Y_1).$$

Given $(E', B', F', p'), (E'', B'', F'', p'')$ Serre fibrations we can consider the product

$$(E' \times E'', B' \times B'', F' \times F'', p' \times p'')$$

and now construct the spectral sequence associated to the bi-cosimplicial R-algebras $Hom(RSin_{*,*}(\pi), R)$ for $\pi = p', p'', p' \times p''$.

We therefore have a map, unique up to homotopy

$$\alpha : Hom(RSin_{*,*}(\pi), R) \otimes Hom(RSin_{*,*}(\pi), R) \to Hom(RSin_{*,*}(\pi \times \pi), R)$$

which we can make explicit via the Alexander-Whitney map in both directions. The filtered complex has the structure of a filtered differential graded algebra.

Via the diagonal map

$$\begin{tikzcd} E \arrow[r, "\Delta"] \arrow[d, "\pi"'] & E \times E \arrow[d, "\pi \times \pi"] \\ B \arrow[r, "\Delta"'] & B \times B \end{tikzcd}$$

the Alexander-Whitney homomorphism therefore induces a product in $Hom(RSin_{*,*}(\pi), R)$ which induces the cup product on $H^*(E)$ and which, being compatible with the filtration, determines a product on the cohomology spectral sequence for every page such that

$$d_r(\alpha_1 \otimes \alpha_2) = (d_r\alpha_1) \otimes \alpha_2 + (-1)^{|\alpha_1|}\alpha_1 \otimes d_r\alpha_2.$$

Chapter 3

Applications of the Serre spectral sequence

Let us now see some applications.
• $SU(n)$.
Let us return to what was already seen in homology.

$$H^*(SU(n);\mathbb{Z}) \cong H^*(S^3 \times S^5 \times \cdots \times S^{2n-1};\mathbb{Z}).$$

We prove by induction that $H^*(SU(n);\mathbb{Z}) = \bigwedge[x_3, \ldots, x_{2n-1}]$.
We observe the fibration $SU(n) \to SU(n+1) \to S^{2n+1}$.
The study of the spectral sequence leads us to conclude that $E_\infty \cong \bigwedge[x_3, \ldots, x_{2n+1}]$. Since by Serre's theorem we know that E_∞ is the graded

$$GH^*(SU(n+1)) = \bigoplus_i F_i H^*(SU(n+1))/F_{i+1}H^*(SU(n+1))$$

it follows that the preimages of the classes $X_3, \ldots, X_{2n+1}$ generate $H^*(SU(n+1))$ which is therefore a quotient of the free integral exterior algebra on $x_3, \ldots, x_{2n+1}$.
Having the same rank it is precisely isomorphic to $\bigwedge[x_3, \ldots, x_{2n+1}]$ and we have concluded.
• $\mathbb{CP}^n$.
We observe the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and the associated cohomological Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{CP}^n; H^q(S^1)) = \begin{cases} H^p(\mathbb{CP}^n) & \text{q=0, 1} \\ 0 & \text{otherwise} \end{cases}$$

We know that E_∞ describes $H^*(S^{2n+1})$. We derive from the spectral sequence $d_2(e \cdot e_k) = e_{k+1}$ and $d_2(e \cdot e_n) = 0$, from which $e_{k+1} = e_k \cdot d_2 e = e_k \cdot e_1 \Rightarrow e_k = e_1^k \Rightarrow H^*(\mathbb{CP}^n) =$

$$\begin{tikzcd}
 & & & \\
e \cdot e_0 \arrow[dr] & e \cdot e_1 \arrow[dr] & e \cdot e_2 \arrow[dr] & \cdots \\
e_0 & e_1 & e_2 & \cdots
\end{tikzcd}$$

$\mathbb{Z}[\overbrace{e_1}^{dim=2}]/(e_1^n)$.

- ΩS^n.

Consider the fibration $\Omega S^n \to PS^n \to S^n$ and look at the cohomological Serre spectral sequence (recalling that PS^n is contractible). To write the E_2 page, we use the Universal Coefficient Theorem and the fact that the non-zero homologies of ΩS^n are distant from each other.

Let us first take n odd greater than 1.

$$\begin{tikzcd}
\cdots & & \\
2n-2 & \mathbb{Z}_{a_2} \arrow[dr] & \mathbb{Z}_{a_2 x} \\
n-1 & \mathbb{Z}_{a_1} \arrow[dr] & \mathbb{Z}_{a_1 x} \\
0 & \mathbb{Z}_1 & \mathbb{Z}_x \\
 & 0 & n
\end{tikzcd}$$

$$d_n(a_1) = x$$

$$d_n(a_k) = a_{k-1}x$$

$$d_n(a_1^k) = k a_1^{k-1} x$$

from which, by induction on k, it follows that

$$a_1^k = k! a_k$$

and therefore

$H^*(\Omega S^n; \mathbb{Z}) = \Gamma_{\mathbb{Z}}[a] =< 1, a, \frac{a^2}{2}, \dots, \frac{a^n}{n}, \dots >\subset \mathbb{Q}[a]$.

When n is even, however, $a_1^2 = 0$ holds.

It can be shown that $a_2^k = k! a_{2k}$ and that $a_1 a_{2k} = a_{2k+1}$:

$$d(a_1 a_{2k}) = x a_{2k} - a_1 \overbrace{a_{2k-1} x}^{=0 \text{ by ind.}} = x a_{2k}$$

and therefore

$$d(a_1 a_{2k}) = d(a_{2k+1}) \Rightarrow a_1 a_{2k} = a_{2k+1}.$$

In this case too we can explicitly describe

$$H^*(\Omega S^n) = \bigwedge\nolimits_{\mathbb{Z}}[a] \otimes \Gamma_{\mathbb{Z}}[b]$$

with a in dimension $n-1$ and b in dimension $2n-2$.
We have exhibited examples of cohomology ring calculations using the Serre spectral sequence, let us now make some considerations of a general nature.
Let $\pi : E \to B$ be a Serre fibration. Assume B and F are path-connected and that the local system $H_*(\pi^{-1})$ is trivial.
Consider the spectral sequence

$$E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E),$$

$$E^2_{n,0} \cong H_n(B).$$

Since no differential arrives at row 0 we have

$$E^{r+1}_{n,0} = \ker(d^r : E^r_{n,0} \to E^r_{n-r,r-1})$$

which is trivial for $r > n$. Therefore

$$E^2_{n,0} \supset E^3_{n,0} \supset \cdots \supset E^{n+1}_{n,0} = E^\infty_{n,0}.$$

Approximating $E \to B$ with CW-complexes and cellular maps we have

$$E^{(n)} \subset \pi^{-1}(B^{(n)})$$

from which it follows that $H_n(E) = \mathrm{Im}(H_n(\pi^{-1}(B^{(n)})) \to H_n(E)) = F_n H_n(E)$ that is, $F_n H_n(E) = H_n(E)$ holds and therefore we can look at the composition

$$H_n(E) = F_n H_n(E) \twoheadrightarrow F_n H_n(E)/F_{n-1}H_n(E) = E^{n+1}_{n,0} \hookrightarrow E^n_{n,0} \hookrightarrow \cdots \hookrightarrow E^2_{n,0} = H_n(B).$$

Proposizione 3.0.1. *The map $H_n(E) \to H_n(B)$ just described coincides with $H_n(\pi)$.*

Proof. By naturality and confronting spectral sequences coming from

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & * \arrow[d] \\
E \arrow[r, "\pi"] \arrow[d, "\pi"] & B \arrow[d, "Id_B"] \\
B \arrow[r, "Id_B"] & B
\end{tikzcd}$$

so one obtains

$$\begin{tikzcd}
H_n(E) \arrow[r, "\text{speciale}"'] \arrow[d, "\pi_*"] & H_n(B) \arrow[d, "Id_{B*}"'] \\
H_n(B) \arrow[r, "Id_{B*}"] & H_n(B)
\end{tikzcd}$$

and it only remains to verify that the bottom map is actually our 'special' map. But this is because the fibration on the right is trivial and the spectral sequence converges at page 2. $\square$

Given a section $B \to E$ the same reasoning tells us that the special map $H_n(E) \to H_n(B)$ has a section. In particular we can conclude that row 0 survives until the term E^∞. Similarly, we can note that $E^2_{0,n} = H_n(F)$ and therefore taking successive quotients

$$H_n(F) = E^2_{0,n} \twoheadrightarrow E^3_{0,n} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{0,n} = F_0 H_n(E) \hookrightarrow H_n(E).$$

Proposizione 3.0.2. *The map $H_n(F) \to H_n(E)$ just described is precisely the map induced by the inclusion $i : F \to E$ in homology.*

Proof. Same as before but using the diagram

$$\begin{tikzcd}
F \arrow[r, "Id_F"'] \arrow[d] & F \arrow[d, "i"] \\
F \arrow[r, "i"] \arrow[d] & E \arrow[d, "\pi"] \\
* \arrow[r] & B
\end{tikzcd}$$

and confronting spectral sequences. $\square$

3.1 Transgression

Consider $E \xrightarrow{\pi} B$ a Serre fibration with F, B path-connected and a trivial local system.

Definizione 3.1.1. Let us consider the differentials in the Serre spectral sequence

$$d^n : E^n_{n,0} \to E^n_{0,n-1}.$$

This homomorphism (and its analogue in cohomology) is called *transgression*. The elements that survive up to the domain of the transgression are called *transgressive*.

For $n = 2$ we have $d^2 : \overbrace{E^2_{2,0}}^{H_2(B)} \to \overbrace{E^2_{0,1}}^{H_1(F)}$, but in general, as seen before, $E^n_{n,0}$ is a submodule of $H_n(B)$ while $E^n_{0,n-2}$ is a quotient of $H_{n-1}(F)$.
Consider the exact sequence of the pair

$$\cdots \to H_m(F) \to H_m(E) \to H_m(E, F) \xrightarrow{\partial_*} H_{n-1}(F) \to \dots$$

and the projection

$$\pi_* : H_n(E, F) \to H_n(B, *).$$

Teorema 3.1.1. *The transgression in the homology Serre spectral sequence coincides with the composition*

$$H_n(B) \to H_n(B,*) \xrightarrow{\pi_*^{-1}} H_n(E,F) \xrightarrow{\partial_*} H_{n-1}(F)$$

and similarly in cohomology

$$\cdots \to H^{n-1}(F) \xrightarrow{\partial^*} H^n(E,F) \xrightarrow{(\pi^*)^{-1}} H^n(B,*) \to H^n(B)$$

Proof. Let us consider the homology case. The other is analogous.
We approximate B with a CW-complex with a single 0-cell.
Therefore every element of $E^n_{n,0}$ is represented by a chain $c \in C_n(\pi^{-1}(B^{(n)})) \subset C_n(E)$ with boundary in $C_{n-1}(\pi^{-1}(B^{(n)})) \subset C_{n-1}(F)$. That is, c is a relative cycle of $C_n(\pi^{-1}(B^{(n)}), F)$.
As we saw in the proposition identifying row 0 of the spectral sequence, the identification of $E^n_{n,0}$ with $H_n(B)$ occurs via the map that sends

$$E^n_{n,0} \to \overbrace{H_n(\pi^{-1}(B^{(n)}), F)}^{Z^n_{n,0}} \ni [c] \xrightarrow{\pi_*} [\pi_*(c)] \in H_n(B).$$

Furthermore the differential $d^n_{n,0} : E^n_{n,0} \to E^n_{0,n-1}$ sends

$$H_n(\pi^{-1}(B^{(n)}), F) \ni [c] \to [\partial c] \in C_{n-1}(F)$$

thus the thesis follows. □

3.2 Serre Classes

Let X be a path-connected topological space such that $\bar{H}_*(X)$ is of torsion. Equivalently, one can say that $\bar{H}_*(X;\mathbb{Q}) = \bar{H}_*(pt,\mathbb{Q})$. Can we say something about the homotopy groups? Are they all torsion?
And what if $\bar{H}_*(X)$ is entirely $p-$ torsion, can we say the same about $\pi_*(X)$?
And what if $H_*(X)$ are all finitely generated?

Definizione 3.2.1. A class $\mathcal{C}$ of abelian groups is called a *Serre class* if $0 \in \mathcal{C}$ and for every short exact sequence

$$0 \to A \to B \to C \to 0$$

it holds that $A, B \in \mathcal{C} \Leftrightarrow B \in \mathcal{C}$.

We observe that a Serre class is closed under

- isomorphisms;
- subgroups;
- quotients;
- extensions: given $A \to B \to C$ exact at B, then if $A, C \in \mathcal{C}$ also $\ker(B \to C), \operatorname{coker}(B \to C)$ are in $\mathcal{C}$ and therefore

$$0 \to \ker(B \to C) \to B \to \operatorname{coker}(B \to C) \to 0$$

is short exact with ends in $\mathcal{C} \Rightarrow B \in \mathcal{C}$.

Let us list a few Serre classes (the proof is an exercise):

- the class of finite abelian groups $\mathcal{C}_{fin}$;
- the class of finitely generated abelian groups $\mathcal{C}_{fg}$;
- the class of torsion abelian groups $\mathcal{C}_{tor}$;
- the class of p-torsion abelian groups.

Definizione 3.2.2. Let $\mathcal{P}$ be a set of primes. Let $\mathcal{C}_{\mathcal{P}}$ be the class of torsion groups such that if $\exists p \in \mathcal{P}$ which divides the order of $A \in \mathcal{C}_{\mathcal{P}}$ then $a = 0$.
We will write $\mathcal{C}_p$ to denote $\mathcal{C}_{\{p\}}$.
This is a Serre class for every prime p.

Let $\mathbb{Z}_{\mathcal{P}}$ be the localization of $\mathbb{Z}$ at the complement of the union of the ideals generated by the primes $(p), p \in \mathcal{P}$. Then $A \in \mathcal{C}_{\mathcal{P}} \Leftrightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathcal{P}} = 0$ therefore in particular

$$A \in \mathcal{C}_p \Leftrightarrow A \otimes \mathbb{Z}_p = 0$$

(we will omit the subscript on the tensor when it is clear that we are working with $\mathbb{Z}$ modules).
We also observe that the intersection of Serre classes is a Serre class.
Let us now give a definition that will allow us to work *modulo a Serre class*.

Definizione 3.2.3. Given a Serre class $\mathcal{C}$, we will say that $A = 0 \mod \mathcal{C}$ if $A \in \mathcal{C}$. Given a morphism $f : A \to B$ we will say

- it is a monomorphism $\mod \mathcal{C}$ if $\ker f \in \mathcal{C}$;
- it is an epimorphism $\mod \mathcal{C}$ if $\operatorname{coker} f \in \mathcal{C}$;
- it is an isomorphism $\mod \mathcal{C}$ if it is both.

Proposizione 3.2.1. *Let $\mathcal{C}$ be a Serre class. The classes of monomorphisms, epimorphisms, isomorphisms $\mod \mathcal{C}$ are closed under composition. The class of isomorphisms respects the "2 out of 3" rule, i.e., if two among $\alpha, \beta, \alpha \circ \beta$ are iso ($\mod \mathcal{C}$) the third one is too.*

Proof. The proof follows from the exactness of the exterior path on the following hexagon

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & & & 0 & \\
 & & \ker\beta \arrow[rr] \arrow[dr] & & \text{coker }\alpha \arrow[ur] \arrow[ddr] & & \\
 & & & B \arrow[ur] \arrow[dr, "\beta"] & & & \\
0 \arrow[r] & \ker\beta\circ\alpha \arrow[r] \arrow[uur] & A \arrow[ur, "\alpha"] \arrow[rr, "\beta\circ\alpha"'] & & C \arrow[r] \arrow[d] & \text{coker }\beta\circ\alpha \arrow[r] \arrow[dl] & 0 \\
 & & \ker\alpha \arrow[u] \arrow[ul] & 0 \arrow[l] & \text{coker }\beta \arrow[l] \arrow[d] & & \\
 & & 0 \arrow[u] & & 0 & &
\end{tikzcd}
$$

□

Let us proceed with some observations.
Let C_* be a chain complex. If $C_n \in \mathcal{C}$ then $H_n(C_*) \in \mathcal{C}$ as well.
Let F be a filtration on A. If $A \in \mathcal{C}$ then $gr_s A \in \mathcal{C}\ \forall\ s$.
If F is finite then $gr_s A \in \mathcal{C}\ \forall s \Leftrightarrow A \in \mathcal{C}$.
Let $\{E^r_{s,t}\}$ be a spectral sequence. If $E^2_{s,t} \in \mathcal{C}\ \forall s,t$ then all other pages (with finite index) are in $\mathcal{C}$. If moreover $\{E^r\}$ is in the first quadrant, then the ∞ page is as well.
If the first-quadrant spectral sequence is induced by a filtration on a complex C_* and $E^2_{s,t} \in \mathcal{C}$ for $s+t=n$ then $H_n(C_*) \in \mathcal{C}$.
As an example of this last consideration, we observe that given a pair (X,A), if two among $\bar{H}_n(A), \bar{H}_n(A), \bar{H}_n(X,A)$ are in $\mathcal{C}$, then the third one is too.
An **IMPORTANT FACT** holds (left as an exercise): the 5-lemma $\mod \mathcal{C}$ continues to hold looking at exact sequences $\mod \mathcal{C}$.

Definizione 3.2.4. A Serre class is called a *Serre ring* if $A, B \in \mathcal{C} \Rightarrow A \otimes B$ and $A * B$ are in $\mathcal{C}$.

Definizione 3.2.5. A Serre class is called a *Serre ideal* if $A \in \mathcal{C} \Rightarrow A \otimes B$ and $A * B$ are in $\mathcal{C}$ for every abelian group B.

All the examples above are Serre rings. The examples without the hypothesis that the groups are finitely generated are Serre ideals.
Let $\mathcal{C}$ be a Serre class and $A \in \mathcal{C}$. We consider the classifying space $BA = K(A,1)$.
We know that $H_1(K(A,1)) = A \in \mathcal{C}$.

Definizione 3.2.6. A Serre ring is called *acyclic* if it contains the homologies of all its classifying spaces.

Let us verify the acyclicity of some Serre classes.

- $\mathcal{C}_{fin}$ is acyclic. By Kunneth it suffices to consider $H_*(BC_n)$ where $C_n = \mathbb{Z}/n\mathbb{Z}$. $C_n \subset S^1$ acts on $S^\infty \subset \mathbb{C}^\infty$. A model of a classifying space is the infinite lens space. In particular we have the fibration

$$S^1 = C_n \backslash S^1 \to \ BC_n \to \mathbb{CP}^\infty$$

Let us study the associated Serre spectral sequence (we report only the cohomological one):

$$\begin{tikzcd}
1 & e \arrow[dr] & ex \arrow[dr] & ex^2 \arrow[dr] & \cdots \\
0 & 1 & x & x^2 & x^3 \\
 & 0 & 2 & 4 & 6
\end{tikzcd}$$

$$E_2^{s,t} = H^s(\mathbb{CP}^\infty) \otimes H^t(S^1)$$

$$H_1(BC_n) = C_n,\ H_2(BC_n) = 0 \Rightarrow H^2(BC_n) = C_n$$

$$\Rightarrow d_2 e = nx \Rightarrow d_2 ex^i = nx^{i+1}$$

We then derive $H^*(BC_n) = \mathbb{Z}[x]/(nx)$ with x in degree 2. Therefore $H_i(BC_n)$ and $H^i(BC_n)$ are finite and $\mathcal{C}_{fin}$ is acyclic.

- Since every torsion group is a direct limit of its finite subgroups, we can conclude that $A \in \mathcal{C}_{tor} \Rightarrow \bar{H}_q(K(A,1)) \in \mathcal{C}_{tor}$, that is, $\mathcal{C}_{tor}$ is acyclic.

- Similarly, the classes of finite $p-$groups and $\mathcal{C}_p$ are acyclic.

- Observing that $H^*(K(\mathbb{Z},1)) = H^*(S^1) = \bigwedge[e]$ with e in degree 1, we conclude that $\mathcal{C}_{fg}$ is acyclic.

Let us make an important observation. Let $\mathcal{C}$ be a Serre ideal; if $H_n(X), H_{n-1}(X)$ are zero mod $\mathcal{C}$ then $H_n(X;M) = 0 \mod \mathcal{C}$ for every abelian group M.
In the case where $\mathcal{C}$ is only a Serre ring, the observation holds for $M \in \mathcal{C}$.

Proposizione 3.2.2. *Let $\pi : E \to B$ be a Serre fibration with B, F path-connected and trivial action $\pi_1(B) \curvearrowright H_*(F)$. Let $\mathcal{C}$ be a Serre ideal.*
If $H_t(F) \in \mathcal{C}\ \ \forall t > 0$ then $\pi_* : H_*(E) \to H_*(B)$ is an isomorphism $\mod \mathcal{C}$.

Proof. By the Universal Coefficient Theorem it holds that

$$E^2_{s,t} = H_s(B, H_t(F)) \in \mathcal{C}$$

Therefore $E^r_{s,t}$ is also in $\mathcal{C}$ for $t > 0$ and similarly for the $E^\infty_{\bullet,\bullet}$ page.
It follows that the map

$$\pi_* : H_*(E) \to H_*(B)$$

is an isomorphism modulo $\mathcal{C}$. □

Proposizione 3.2.3. *Let $\pi : E \to B$ be such that F, B are path-connected and $\pi_1(B) = 0$. Let $\mathcal{C}$ be a Serre ring such that*

- *$H_s(B) \in \mathcal{C}$ for every $0 < s < n$;*

- *$H_t(F) \in \mathcal{C}$ for every $0 < t < n-1$ **or** $\mathcal{C}$ is a Serre ideal.*

Then $\pi_* : H_i(E,F) \to H_i(B, b_0)$ is an isomorphism $\mod \mathcal{C}$ for $i \leqslant n$.

Proof. We use the homological Serre spectral sequence in the relative case.

$$E^2_{s,t} = H_s(B, b_0; H_t(F)) \Rightarrow H_{s+t}(E, F).$$

For $s = 0$ and $s = 1$ we have $E^2_{s,t} = 0$ (we are using that B is path-connected and simply connected).
Furthermore $E^2_{s,t} \in \mathcal{C}$ for (s,t) in $[2, n-1] \times [1, n-2]$ therefore for $s + t \leqslant n$ the only non-zero groups modulo $\mathcal{C}$ are $E^2_{i,0} = H_i(B, b_0)$.
We therefore derive that $\pi_* : H_i(E, F) \to H_i(B, b_0)$ is an isomorphism modulo $\mathcal{C}$. $\square$

Let us now discuss an important theorem.

Teorema 3.2.4 (Hurewicz Theorem modulo an acyclic Serre ring)**.** *Let $\mathcal{C}$ be an acyclic Serre ring. Let X be simply connected and $n \geqslant 2$.*

$$\pi_q(X) \in \mathcal{C} \ \ \forall q < n \Leftrightarrow \bar{H}_q \in \mathcal{C} \ \ \forall q < n$$

and in this case the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism modulo $\mathcal{C}$.

Before the proof, let us show some corollaries (based on the acyclicity of some rings verified previously).

- $H_q(X)$ is finitely generated for every $q < n$ if and only if $\pi_q(X)$ is finitely generated for every $q < n$;
- For p prime, $H_q(X)$ is of $p-$torsion $\forall\ q < n$ if and only if $\pi_q(X)$ is of $p-$torsion for every $q < n$;
- $\bar{H}_q(X; \mathbb{Q}) = 0$ for every $q < n$ if and only if $\pi_q(X) \otimes \mathbb{Q} = 0$ for every $q < n$ and

  $$h : \pi_n(X) \otimes \mathbb{Q} \to H_n(X; \mathbb{Q})$$

  is an iso.

Let us proceed to the proof.

Proof. We retrace with some variations the proof of the Hurewicz theorem.
We proceed by induction on n, using the fibration $\Omega X \to PX \to X$ and show that if $\pi_q(X) \in \mathcal{C}$ for $q < n$ then $\pi_q(X) \to H_q(X)$ is an isomorphism modulo $\mathcal{C}$.
Base step: $n = 2$. By the Hurewicz theorem $\pi_2(X) \to H_2(X)$ is an isomorphism.
Inductive step: Consider the diagram

$$\begin{tikzcd}
\pi_q(X, x_0) \arrow[d, "h"] & \pi_q(PX, \Omega X) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "h"] & \pi_{q-1}(\Omega X) \arrow[d, "(2)"', "h"] \\
H_q(X, x_0) & H_q(PX, \Omega X) \arrow[l, "(1)"'] \arrow[r, "\cong"'] & H_{q-1}(\Omega X)
\end{tikzcd}$$

and let us verify that $(1), (2)$ are isomorphisms modulo $\mathcal{C}$.
If $\pi_2(X) = 0$ (equivalently $\pi_1(\Omega X) = 0$, which we cannot assume a priori), then (2) is an isomorphism by the induction hypothesis (using ΩX instead of X).

Furthermore, using the proposition on the fibration with B simply connected and $\mathcal{C}$ a Serre ring, since again by the induction hypothesis $H_i(X, x_0) \in \mathcal{C}$ for $i < n$, we can conclude that (1) is an isomorphism modulo $\mathcal{C}$.
If $\pi_2(X) \neq 0$ we proceed as follows: let us look at the Whitehead tower

$$\begin{tikzcd} K = K(\pi_2(X), 1) \arrow[d] \\ Y \arrow[d] \\ X \end{tikzcd}$$

Since $\pi_2(X) \in \mathcal{C}$ we have $H_i(K) \in \mathcal{C}$ for all positive i, by the hypothesis of acyclicity of $\mathcal{C}$. It follows, looking at the long exact sequence of the pair (Y, K), that $H_i(Y, y_0) \to H_i(Y, K)$ is an isomorphism modulo $\mathcal{C}$.
Since X is simply connected we can apply again the proposition on the fibration with simply connected base and Serre ring to the map $(Y, K) \to (X, x_0)$.

We obtain

$$H_i(Y, K) \to H_i(X, x_0)$$

isomorphism modulo $\mathcal{C}$ for $i \leqslant n$. Therefore $H_i(Y, y_0) \to H_i(X, x_0)$ is an isomorphism modulo $\mathcal{C}$ for $i \leqslant n$.
The map $\pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \geqslant 3$ and $\pi_2(Y) = \pi_1(Y) = 0$, thus one can apply the induction hypothesis to Y and conclude. □

We exhibit a further corollary (which we phrase as a proposition).

Proposizione 3.2.5. *Let X be simply connected, p a prime and $n \geqslant 2$. Then*

$$\pi_i(X) \otimes \mathbb{Z}_{(p)} = 0 \ \forall i < n \Leftrightarrow \bar{H}_i(X; \mathbb{Z}_{(p)}) = 0 \ \forall i < n$$

and in this case the map $h : \pi_n(X) \otimes \mathbb{Z}_{(p)} \to \bar{H}_n(X; \mathbb{Z}_{(p)})$ is an isomorphism.

Proof. It suffices to use the Hurewicz theorem modulo $\mathcal{C}$ taking $\mathcal{C}_p$ as the Serre ring. □

We also give a relative version of the theorem.

Teorema 3.2.6 (Relative Hurewicz modulo $\mathcal{C}$)**.** *Let $\mathcal{C}$ be an acyclic Serre* ***ideal*** *(note the stronger hypothesis). Let (X, A) be a pair of simply connected topological spaces and $n \geqslant 2$. Then*

$$\pi_i(X, A) \in \mathcal{C} \ \ \forall 2 \leqslant i < n \Leftrightarrow H_i(X, A) \in \mathcal{C} \ \ \forall 2 \leqslant i < n$$

and in this case $h : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism modulo $\mathcal{C}$.

Idea. Let F be the homotopy fiber of $A \hookrightarrow X$. Then we obtain

$$\begin{tikzcd} \pi_n(X, A) \arrow[d, "h"] & \pi_n(PX, F) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "h"'] & \pi_{n-1}(F) \arrow[d, "h", "\cong"'] \\ H_n(X, A) & H_n(PX, F) \arrow[l, "p*"'] \arrow[r, "\cong"] & H_{n-1}(F, f_0) \end{tikzcd}$$

and to prove that p_* is an isomorphism modulo $\mathcal{C}$ we need the Serre spectral sequence for

$$\begin{tikzcd} \Omega X \arrow[r, hook] & (PX, F) \arrow[d] \\ & (X, A) \end{tikzcd}$$

with $E^2_{s,t} = H_s(X, A; H_t(\Omega X))$. At this point we use the fact that $\mathcal{C}$ is a Serre ideal to apply one of the previous propositions and conclude that p_* is an isomorphism modulo $\mathcal{C}$. □

Teorema 3.2.7 (Whitehead Theorem modulo $\mathcal{C}$)**.** *Let $\mathcal{C}$ be an acyclic Serre ideal and $f : X \to Y$ a map of simply connected spaces. Let $n \geqslant 2$. The following are equivalent*

- *$f_\# : \pi_i(X) \to \pi_i(Y)$ is an isomorphism modulo $\mathcal{C}$ for $i \leqslant n-1$, an epimorphism modulo $\mathcal{C}$ for $i = n$;*
- *$f_* : H_i(X) \to H_i(Y)$ is an isomorphism modulo $\mathcal{C}$ for $i \leqslant n-1$ and an epimorphism for $i = n$.*

Proof. As always, we can suppose that f is an inclusion by replacing Y with M_f. We now have the diagram

$$\begin{tikzcd} \dots \arrow[r] & \pi_{n+1}(Y,X) \arrow[r] \arrow[d, dashed, "h"] & \pi_n(X) \arrow[r, "f_\#"] \arrow[d, dashed, "h"] & \pi_n(Y) \arrow[r] \arrow[d, dashed, "h"] & \pi_n(Y,X) \arrow[d, dashed, "h"] & \dots \\ \dots \arrow[r] & H_{n+1}(Y,X) \arrow[r] & H_n(X) \arrow[r, "f_*"'] & H_n(Y) \arrow[r] & H_n(Y,X) & \dots \end{tikzcd}$$

Therefore the first statement is equivalent to $\pi_i(Y, X) = 0 \mod \mathcal{C}$ for $i \leqslant n$ and by Hurewicz this is equivalent to $H_i(Y, X) = 0 \mod \mathcal{C}$, which is equivalent to the second statement. □

Proposizione 3.2.8. *Let X, Y be topological spaces such that $H_q(\bullet, \mathbb{Z}_{(p)})$ is finitely generated for every q.*
If $f : X \to Y$ is an isomorphism in $H_q(\bullet, \mathbb{Z}/p\mathbb{Z})$ for every q then it induces an isomorphism modulo $\mathcal{C}_p$ in $H_q(\bullet, \mathbb{Z})$.

Proof. We recall that $A \xrightarrow{\alpha} B$ is an isomorphism modulo $\mathcal{C}_p$ if and only if $A \otimes \mathbb{Z}_{(p)} \xrightarrow{\alpha \otimes 1} B \otimes \mathbb{Z}_{(p)}$ is an isomorphism: the module $\mathbb{Z}_{(p)}$ is flat and therefore the exact sequence

$$0 \to \ker \alpha \to A \to B \to \text{coker } \alpha \to 0$$

implies the exactness of

$$0 \to \ker \alpha \otimes \mathbb{Z}_{(p)} \to A \otimes \mathbb{Z}_{(p)} \to B \otimes \mathbb{Z}_{(p)} \to \text{coker } \alpha \otimes \mathbb{Z}_{(p)} \to 0$$

therefore we have an isomorphism between the two central terms if and only if $\ker \alpha, \text{coker } \alpha \in \mathcal{C}_{(p)}$.
A finitely generated module over $\mathbb{Z}_{(p)}$ is the trivial module if it is so after tensorizing with $\mathbb{F}_p$. Then we show that the kernel and cokernel of $f_* : H_*(X) \to H_*(Y)$ are zero when we tensor with $\mathbb{F}_p$.

Let now Z be the mapping cone of f. We know that

$$0 = \bar{H}_*(Z; \mathbb{Z}/p\mathbb{Z})$$

by hypothesis and therefore $H_*(C_f, X; \mathbb{Z}/p\mathbb{Z}) = 0$.
Since $\mathbb{Z}_{(p)}$ is Noetherian, the hypotheses of finite generation of the homologies (with localized coefficients) of X and Y guarantee the finite generation of $H_q(Z; \mathbb{Z}_{(p)})$ for every q.
From the Universal Coefficient Theorem we have the embedding $\bar{H}_*(Z; \mathbb{Z}_{(p)}) \otimes \mathbb{F}_p \hookrightarrow H_*(Z; \mathbb{F}_p)$, from which $\bar{H}_*(Z; \mathbb{Z}_{(p)}) = 0$ therefore $f_* \otimes 1$ is an isomorphism, which is the thesis. □

Proposizione 3.2.9. *Let X, Y be simply connected with homology with coefficients in $\mathbb{Z}_{(p)}$ finitely generated. Suppose a morphism f induces isomorphisms between the homologies with coefficients in $\mathbb{Z}/p\mathbb{Z}$, then $f_\# : \pi_*(X) \otimes \mathbb{Z}_{(p)} \to \pi_*(Y) \otimes \mathbb{Z}_{(p)}$ is an isomorphism.*

Proof. It follows from the previous proposition and from the Whitehead theorem modulo $\mathcal{C}$. □

3.3 Serre's Theorem

We proceed preliminarily to calculate the homology of $K(A, n)$.
If $A \in \mathcal{C}_{tor}$ the Hurewicz theorem modulo $\mathcal{C}$ tells us that $\bar{H}_*(K(A, n); \mathbb{Q}) = 0$ as well.
For $K(\mathbb{Z}, 1) \cong S^1$ we have $H^*(K(\mathbb{Z}, 1), \mathbb{Q}) = \bigwedge_{\mathbb{Q}}[i_1]$ with i_1 a generator in degree 1.
For $K(\mathbb{Z}, 2) \cong \mathbb{CP}^\infty$ we can calculate the homology as follows.
We use the fibration $K(\mathbb{Z}, n-1) \to PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ and observe the Serre spectral sequence in cohomology (for $n = 2$).

$$\begin{array}{c|ccccc}
1 & i_1 & & i_1 \cdot i_2 & & \dots \\
0 & 1 & & i_2 & & {i_2}^2 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

$$d(i_1) = i_2$$

$$d(i_1 \cdot i_2^k) = i_2^{k+1}$$

da cui $H^*(K(\mathbb{Z}, 2); \mathbb{Q}) = \mathbb{Q}[i_2]$ con i_2 di grado 2.
Procediamo adesso con lo spazio $K(\mathbb{Z}, 3)$ in maniera simile.

$$\begin{array}{c|cc} & \dots & \\ 4 & i_2^2 & \\ 2 & i_2 & \\ 0 & 1 & i_3 \\ \hline & 0 & 3 \end{array}$$

we obtain $d_3(i_2) = i_3, d_3(i_2^k) = k \cdot i_2^{k-1} \cdot i_3$

$$\Rightarrow H^*(K(\mathbb{Z},3);\mathbb{Q}) = \bigwedge[i_3], \;\; |i_3| = 3.$$

For the general case of $K(\mathbb{Z}, n)$ we can proceed inductively (on n). Using the fibration

$$K(\mathbb{Z}, n-1) = \Omega K(\mathbb{Z}, n) \to PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$$

we are able to obtain (via the Serre spectral sequence)

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \bigwedge[i_n] & \text{for } n \text{ odd} \\ \mathbb{Q}[i_n] & \text{for } n \text{ even} \end{cases}$$

n **odd**

$$\begin{array}{c|cc} 2(n-1) & i_{n-1}^2 & i_{n-1}^2 \cdot i_n \\ n-1 & i_{n-1} & i_{n-1} \cdot i_n \\ 0 & & i_n \\ \hline & 0 & n \end{array}$$

n **even**

$$\begin{array}{c|cccc} n-1 & i_{n-1} & i_n \cdot i_{n-1} \longrightarrow & i_n^2 \cdot i_{n-1} & \\ 0 & 1 & i_n & i_n^2 & \\ \hline & 0 & n & 2n & \dots \end{array}$$

We proceed now by stating an important theorem of Serre.

Teorema 3.3.1 (Serre's Finiteness Theorem)**.** *The homotopy groups $\pi_i(S^n)$ are finite for every i, except $\pi_n(S^n)$ and $\pi_{2n-1}(S^n)$ for n even. In that case, they are finitely generated and have rank* 1.

Proof. For $n = 1$ it is clear, S^1 is an Eilenberg-Maclane space.
Let us assume $n \geqslant 2$ and consider the first non-zero map in the Postnikov tower of S^n, the map $S^n \to K(\mathbb{Z}, n)$.
If n is odd, from the calculation of rational homology above, we know that this map induces isomorphisms in $H_*(\bullet;\mathbb{Q})$. Therefore, by the Whitehead theorem modulo $\mathcal{C}$, it induces them also in rational homotopy $\pi_*(\bullet) \otimes \mathbb{Q}$ and thus the thesis follows.
If n is even, we proceed as follows. Consider the homotopy fiber F (which, we note, is a

homotopy truncation) and let us calculate its cohomology

$$
\begin{tikzcd}
2n-1 & i_{2n-1} \arrow[rrd] & i_n \cdot i_{2n-1} \arrow[rrd] & i_n^2 \cdot i_{2n-1} & i_n^3 \cdot i_{2n-1} & \cdots \\
 & 1 & i_n & i_n^2 & i_n^3 & \cdots \\
 & 0 & n & 2n & 3n &
\end{tikzcd}
$$

from here it follows that F has the same homology and rational homotopy of a $K(\mathbb{Z}, 2n-1)$, and the thesis follows. $\square$