K-theory working group

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January 2026

Abstract

1 Introduction

Let X be a Noetherian scheme. Assume first that X is irreducible. Then, for x the generic point of X, $k(x)$-Mod $= M(\text{Spec } \mathcal{O}_{X,x})$ is the localisation of $M(X)$ at the Serre full subcategory $M^1(X)$ of $M(X)$ whose objects are coherent $\mathcal{O}_X$-modules whose support is of codimension at least one. Irreducibility is not actually necessary: letting X^0 be the set of generic points of X, one has in general that

$$M(X)/M^1(X) \cong \coprod_{x\in X^0} M(\text{Spec}(k(x)))$$

One can then iterate this procedure: letting X^p be the set of codimension p points of X and $M^p(X)$ the full Serre subcategory of $M(X)$ whose objects are $\mathcal{O}_X$-modules whose support have codimension at least p, one gets that $M^{p+1}(X)$ is a Serre subcategory of $M^p(X)$ and $M^p(X)/M^{p+1}(X) \cong \coprod_{x\in X^p} \mathcal{A}(\mathcal{O}_{x,X})$ where $\mathcal{A}(\mathcal{O}_{x,X})$ is the category of finite length $\mathcal{O}_{x,X}$-modules. The localisation sequence of K-theory thus gives rise to the exact sequences

$$\cdots \to K_i(M^{p+1}(X)) \to K_i(M^p(X)) \to \bigoplus_{x\in X^p} K_i(\mathcal{A}(\mathcal{O}_{x,X})) \to K_{i-1}(M^{p+1}(X)) \to \dots$$

. By dévissage, $K_i(\mathcal{A}(\mathcal{O}_{x,X})) \cong K_i(k(x))$, and thus we have an exact sequence

$$\cdots \to K_i(M^{p+1}(X)) \to K_i(M^p(X)) \to \bigoplus_{x\in X^p} K_i(k(x)) \to K_{i-1}(M^{p+1}(X)) \to \dots$$

In today's session we will see how these sequences naturally give rise to a spectral sequence and view some consequences. We follow [1] p 64 - 74.

2 Exact couples

Exact couples is the tool from homological algebra that will allow us to construct the spectral sequence.

Definition 1. *An exact couple is a diagram in an abelian category*

$$\begin{tikzcd} D \arrow[rr, "b"'] & & D \arrow[dl, "c"] \\ & E \arrow[ul, "a"] & \end{tikzcd}$$

such that any two arrows form an exact sequence.

The following lemma is key to the construction of spectral sequences:

Lemma 1. *Given an exact couple*

$$\begin{tikzcd} D_1 \arrow[rr, "b_1"'] & & D_1 \arrow[dl, "c_1"] \\ & E_1 \arrow[ul, "a_1"] & \end{tikzcd}$$

one can build an exact couple

$$\begin{tikzcd} D_2 \arrow[rr, "b_2"'] & & D_2 \arrow[dl, "c_2"] \\ & E_2 \arrow[ul, "a_2"] & \end{tikzcd}$$

by letting $D_2 = im\ b_1$ and, letting $d_1 = c_1 \circ a_1$, $E_2 = \ker d_1 / im\ d_1$, a_2 the map induced by a_1, b_2 that induced by b_1 and $c_2(b_1(x)) = [c_1(x)]$ where $[y]$ denotes the class of $y \in \ker a_1$ in E_2 (we use elements for convenience, but this can clearly be done in any abelian category). Moreover, repeating the operation, one gets exact couples

$$\begin{tikzcd} D_n \arrow[rr, "b_n"'] & & D_n \arrow[dl, "c_n"] \\ & E_n \arrow[ul, "a_n"] & \end{tikzcd}$$

such that $D_n = im\ b_1^{n-1}$, $E_n = a_1^{-1}(im\ b_1^{n-1})/c_1(\ker b_1^{n-1})$, a_n and b_n are the maps induced respectively by a_1 and b_1, and $c_n(b_1^{n-1}(x)) = [c_1(x)]$.

Proof. Diagram chasing plus induction. □

Spectral sequence come into play when we add grading: assume from now on that E_1 and D_1 are $\mathbb{Z}$-bigraded abelian groups, and that a_1 has bidegree $(1,0)$, b_1 has bidegree $(-1,1)$ and c_1 has bidegree $(0,0)$. In this situation, the data of an exact couple correspond to having exact sequences

$$\cdots \to D_1^{p+1,q} \to D_1^{p,q} \to E_1^{p,q} \to D_1^{p+1,q} \to D_1^{p,q+1} \to \ldots$$

Since $\mathbb{Z}$-bigraded abelian groups form an abelian category, E_n and D_n inherit a stucture of $\mathbb{Z}$-bigraded module, and one easily shows that a_n, b_n, c_n and

$d_n = c_n \circ a_n$ have respective bidegrees $(1,0)$, $(-1,1)$, $(n-1,1-n)$ and $(n,1-n)$ for all n. One thus obtains that $(E_n^{p,q}, d_n)_{n\in\mathbb{N}^*,(p,q)\in\mathbb{Z}^2}$ form a spectral sequence.

In order to insure convergence, one adds the following hypotheses:
-For each n, $D_1^{n-q,q} = 0$ for $q < q_0(n)$ for some $q_0(n) \in \mathbb{Z}$.
-For each n, the map $D_1^{n-q+1,q-1} \to D_1^{n-q}$ induced by b_1 is an isomorphism for $q > q_1(n)$ for some $q_1(n) \in \mathbb{Z}$.

We then let A^n be the inductive limit of the $D_1^{n-q,q}$ and write $F^pA^n = \operatorname{im}(D_1^{p,n-p} \to A^n)$.

Proposition 1. *$(E_n^{p,q}, d_n)$ converges to A^n with respect to this filtration, i.e. for any p,q, for n large enough, $E_n^{p,q} \cong E_\infty^{p,q} \cong F^pA^{p+q}/F^{p+1}A^{p+q}$.*

Proof. Fix $(p,q) \in \mathbb{Z}^2$. The exact sequence associated to the exact couple $(D_n, E_n, a_n, b_n, c_n)$ gives that $D_n^{p-n+2,q+n-2} \to D_n^{p-n+1,q+n-1} \to E_n^{p,q} \to D_n^{p+1,q}$ is exact. We have

$$\begin{aligned} D_n^{p-n+2,q+n-2} &= D_1^{p-n+2,q+n-2} \cap \operatorname{im} b_1^{n-1} \\ &= \operatorname{im}\Big(D_1^{p+1,q-1} \to D_1^{p-n+2,q+n-2}\Big) \\ &\cong \operatorname{im}\Big(D_1^{p+1,q-1} \to F^{p+1}A^{p,q}\Big)\,. \\ &\cong F^{p+1}A^{p+q} \end{aligned}$$

and similarly $D_n^{p-n+1,q+n-1} \cong F^pA^{p+q}$.
Furthermore, $D_n^{p+1,q} = D_1^{p+1,q} \cap \operatorname{im} b_1^{n-1} = \operatorname{im}(D_1^{p+n,q-n-1} \to D_1^{p+1,q}) = 0$ for n large enough since $(p+n)+(q-n-1) = p+q-1$ is fixed and $q-n-1$ goes to $-\infty$. Thus $E_n^{p,q} \cong F^pA^{p+q}/F^{p+1}A^{p+q}$ for n large enough. $\square$

Example 1. *Spectral sequences associated to filtrations of complexes can be seen as special cases of spectral sequences associated to an exact pair: if F^pK is a decreasing filtration of the complex K, then we have exact sequences $0 \to F^{p+1}K \to F^pK \to F^pK/F^{p+1}K \to 0$ which lead to long exact sequences of cohomology and thus to an exact couple*

$$\begin{tikzcd} H^*(F^*K) \arrow[rr, "b"'] & & H^*(F^*K) \arrow[dl, "c"] \\ & H^*(C^*) \arrow[ul, "a"] & \end{tikzcd}$$

where $C^p = F^pK/F^{p+1}K$. In order to have the right bidegrees for a,b,c, one should choose the gradations so that $E_1^{p,q} = H^{p+q}(F^pK)$ and $D_1^{p,q} = H^{p+q}(F^pK/F^{p+1}K)$.

3 The Brown-Gersten-Quillen spectral sequence

Applying the above formalism to the exact sequence we obtained in the introduction, we obtain:

Theorem 1. *Let X be a noetherian scheme. With the notations of the introduction, there is a spectral sequence with first term*

$$E_1^{p,q}(X) = \coprod_{x \in X^p} K_{-p-q}(k(x))$$

such that, assuming X is of finite Krull dimension,

$$E_1^{p,q}(X) \Longrightarrow G_{-p-q}(X)$$

Moreover, the sequence is functorial with respect to flat morphisms and commutes with directed limits provided the transition maps are affine and flat and all terms and the limit are noetherian.

Proof. The construction of the spectral sequence follows from the previous section (put $E_1^{p,q} = \coprod_{x \in X^p} K_{-p-q}(k(x))$ and $D_1^{p,q} = K_{-p-q}(M^p(X))$ with the convention that $X^p = X^0$ and $M^p(X) = M(X)$ for $p \leq 0$ and $K_i = 0$ for $i < 0$), and if X is of finite Krull dimension, then the hypotheses before proposition 1 are verified, and thus it converges to the directed colimit of the $K_{-p-q}(M^n(X))$ with n going to $-\infty$, thus this gives $K_{-p-q}(M(x)) = G_{-p-q}(X)$. For functoriality, we use the going up theorem for flat maps to prove that if $f : X \to Y$ is flat, then all points of the inverse image of a codimension p point have codimension at least p and so f^* maps $M^p(Y)$ to $M^p(X)$ (being of codimension at least p means you can go up p times; by the going up theorem if you can go up p times on Y then any point of your inverse image can go up at least p times on X); thus we have a natural map in our original long exact sequence, and hence a natural map of spectral sequences. We omit the proof of commutation with directed colimits. □

Proposition 2. *Let $\phi : A \to B$ a flat morphism; then ϕ has the going up property, i.e. if $p' \subset p$ are prime ideals of A and q is a prime ideal of B such that $\phi^{-1}(q) = p$, then there exists $q' \subset q$ a prime ideal such that $\phi^{-1}(q') = p'$.*

Proof. One easily proves that it suffices to prove this in the case $p' = 0$, p is the unique maximal ideal; then it amounts to find a non-trivial ideal in $B \otimes_A A_{p'}$, so to prove that $B \otimes_A A_{p'} \neq 0$, which follows from flatness of B since A injects into $A_{p'}$ and $B \otimes_A A = B \neq 0$. □

For the rest of the talk we will focus on consequences the following equivalent properties (*) and cases in which they hold true:

Proposition 3. *Let X be a Noetherian scheme. The following are equivalent:*
(i) For all $p \geq 0$, $M^{p+1}(X) \to M^p(X)$ induces 0 on K-groups.
(ii) For all $q \leq 0$, $E_2^{p,q}(X) = 0$ for $p \neq 0$, and $G_{-q}(X) \to E_2^{0,q}(X)$ is an isomorphism.
(iii) For all $n \geq 0$, we have an exact sequence

$$0 \to G_n(X) \to \bigoplus_{x \in X^0} K_n(k(x)) \to \bigoplus_{x \in X^1} K_{n-1}(k(x)) \to \dots$$

whose maps are induced by d_1.

Proof. This is really a result about graded exact couples and only uses the exact sequence of the introduction. $(ii) \iff (iii)$ is because the sequence in (iii) is given by $(E_1^{p,q}, d_1)$ and $E_2^{p,q}$ is its cohomology. (i) gives that the long exact sequence of the introduction breaks up into short exact sequences

$$0 \to K_i(M^p(X)) \to \bigoplus_{x \in X^p} K_i(k(x)) \to K_{i-1}(M^{p+1}(X)) \to 0$$

which, pieced together, give (iii), since $K_n(M^0(X)) = K_n(M(X)) = G_n(X)$ by definition. For the last direction $(iii) \Rightarrow (i)$, we prove by induction on p that for all i, the exact sequence of the introduction splits into short exact sequences

$$0 \to K_i(M^p(X)) \to \bigoplus_{x \in X^p} K_i(k(x)) \to K_{i-1}(M^{p+1}(X)) \to 0$$

If it is true for $p-1$ and $p-2$, then as the map induced by d_1 is the composite

$$\bigoplus_{x \in X^{p-2}} K_{i+1}(k(x)) \to K_i(M^{p-1}(X)) \to \bigoplus_{x \in X^{p-1}} K_i(k(x))$$

where the first map is an epimorphism by the case $p-2$ and the second map is a monomorphism by the case $p-1$, and hence the image of this map induced by d_1 is $K_i(M^{p-1}(X))$. But the image of the map

$$\bigoplus_{x \in X^{p-2}} K_{i+1}(k(x)) \to \bigoplus_{x \in X^{p-1}} K_i(k(x))$$

is the kernel of

$$\bigoplus_{x \in X^{p-1}} K_i(k(x)) \to \bigoplus_{x \in X^p} K_{i-1}(k(x))$$

by (iii), and thus the composite

$$\bigoplus_{x \in X^{p-1}} K_i(k(x)) \to K_{i-1}(M^p(X)) \to \bigoplus_{x \in X^p} K_{i-1}(k(x))$$

has kernel $K_i(M^{p-1}(X))$. But by induction hypothesis for $p-1$ again,

$$0 \to K_i(M^{p-1}(X)) \to \bigoplus_{x \in X^{p-1}} K_i(k(x)) \to K_{i-1}(M^p(X)) \to 0$$

is exact, so that the kernels of

$$\bigoplus_{x \in X^{p-1}} K_i(k(x)) \to K_{i-1}(M^p(X)) \to \bigoplus_{x \in X^p} K_{i-1}(k(x))$$

and of

$$\bigoplus_{x \in X^{p-1}} K_i(k(x)) \to K_{i-1}(M^p(X))$$

coincide, and $\bigoplus_{x\in X^{p-1}} K_i(k(x)) \to K_{i-1}(M^p(X))$ is an epimorphism by the case $p-1$, so $K_{i-1}(M^p(X)) \to \bigoplus_{x\in X^p} K_{i-1}(k(x))$ is a monomorphism. This holds for all i, and so by the exact sequence of the introduction, we thus have that $\bigoplus_{x\in X^p} K_i(k(x)) \to K_{i-1}(M^{p+1}(X))$ has cokernel the kernel of $K_{i-1}(M^p(X)) \to \bigoplus_{x\in X^p} K_{i-1}(k(x))$, i.e. 0, so we get that the exact sequence of the introduction splits into short exact sequences, which proves (i). ☐

Proposition 4. *If Spec($\mathcal{O}_{x,X}$) verifies (*) for all $x \in X$. Then, denoting by $\mathcal{G}_{n,X}$ the sheafification of $U \to G_n(U)$ for the Zariski topology, we have a flasque resolution*

$$0 \to \mathcal{G}_{n,X} \to \bigoplus_{x\in X^0} (i_x)_* K_n(k(x)) \to \bigoplus_{x\in X^1} (i_x)_* K_{n-1}(k(x)) \to \dots$$

In particular, we have an isomorphism $E_2^{p,q}(X) \cong H^p(X, \mathcal{G}_{-q,X}$ for all $q \leq 0$ as they both compute the cohomology at the pth stage of the above sequence for $n = q$.

Proof. We want to show that the sequence of sheaves described is exact (note that the maps are well defined in any case, no matter whether or not (*) holds). It thus suffices to check exactness when localising at every point x. But G theory commutes with directed colimits of flat affine maps, so we obtain exactly the exact sequence of the previous proposition. ☐

Gersten's conjecture is the statement that $(*)$ holds for any X of the form $X = \operatorname{Spec}(R)$ for R a regular local ring. We now give proves of some special cases.

Proposition 5. *For any field k and positive integer n, $R = k[[x_1, \dots, x_n]]$ verifies $(*)$.*

Proof. We want to show that the maps on K-theory induced by $M^{p+1}(X) \to M^p(X)$ are zero. A module whose support is of codimension $p+1$ has support of codimension p in some hypersurface, so comes from $M^p(\operatorname{Spec}(R/tR))$ for some non invertible $t \in R$. As K-theory commutes with directed limits, it suffices thus to show that $M^p(\operatorname{Spec}(R/tR)) \to M^p(\operatorname{Spec}(R))$ is 0 for any nonzero $t \in R$. By Weierstrass preparation theorem, up to a change of coordinates, denoting $A = k[[x_1, \dots, x_{n-1}]]$, the morphism $A \to R/tR$ is finite injective. Let $B = R \otimes_A R/tR$. Then B is a finite R-module, and we have a natural map $B \to R/tR$. Crucially, the kernel of this map is principal, since it is generated by $(x_n - a)$ for a the image of x_n in R/tR. For any finite R/tR-module M, $0 \to B \otimes_{R/tR} M \to B \otimes_{R/tR} M \to M \to 0$ where the first map is given by multiplication by $(x_n - a)$ is a sequence of finite R-modules (since B is finite over R) and is exact because we have an exact sequence $B \to B \to R/tR \to 0$ (this accounts for all maps except the first), and up to replacing x_n with $x_n - a$ (this makes sense as there is a non-trivial k-morphism sending $x_n - a$ to 0 so a no constant term), we see that since $R \cong A[[x_n]]$, $B \cong (R/tR)[[x_n]]$, so we see by hand that the first map does give an injection. Therefore, denoting by F the functor $M \to M \otimes_{R/tR} R$, we see that it yields a functor $M^p(R/tR) \to M^p(R)$ and that we have an exact sequence $0 \to F \to F \to G \to 0$ where G is restriction of scalars. Hence, on K theory, we have $F_* - F_* + G_* = 0$, i.e. G gives 0 on K-theory, which is what we want. □

Note that this proof can be adapted to R the ring of convergent power series with coefficients in a field with an absolute value since Weierstrass preparation continues to hold.

Proposition 6. *(Quillen) If R is a regular semi-local ring which is a localisation of a finitely generated k-algebra for some field k, then $(*)$ holds.*

Proof. We give a sketch of how the proof works in the case R is smooth and k is infinite. The argument resembles that of the previous proof. Here we can write R as a localisation of a finitely generated k-algebra A, and up to localising it on an affine open subset, assume A is smooth. Write R as the localisation of A at S for S a finite set of primes of A. By arguments similar to those given for the previous proposition, it suffices to prove that for all $t \in A$ not a zero divisor for $f \in A$ not vanishing at any element of S, eventually $M(A/tA) \to M(A_f)$ gives 0 on K-theory (the set of such f is a directed poset with respect to the relation of divisibility and we have corresponding factorisations of the morphisms $M(A/tA) \to M(A_f) \to M(A_{f'})$ when $f|f'$, so eventually means eventually in this poset). By Noether normalisation plus Bertini theorem (for the details, see Srinivas' book), there exists a subring B of A with $B \cong k[x_1, \dots, x_{r-1}]$ such that A/tA is finite over B and A is smooth over B at the points of S. Letting $A' = A \otimes_B A/tA$[1], we have that $A \to A'$ is finite since $B \to A/tA$ is finite, and we have a canonical map $A' \to A/tA$. If we can show that the kernel of this map is principal after localising enough, we can win by applying an argument similar to the one at the end of the proof of the previous proposition. Let S' be the inverse image of S by $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ (which is finite since this map is finite). Since the relative dimension of A' over A/tA is one (because $B \to A/tA$ is finite and the relative dimension of A over A/tA is one), this kernel is indeed locally principal at elements of S' (the corresponding geometric fact is that a codimension 1 subvariety of an algebraic variety is locally a hypersurface). Hence, choosing f far enough in the poset, we may assume that the kernel of $A'_f \to A/tA$ is principal. Moreover, since A is smooth over B at S, A' is smooth over A/tA at S', and choosing f far enough in the poset, we may also assume A'_f is smooth over A/tA, so in particular flat. Hence for any finite A/tA-module M, $0 \to M \otimes_{A/tA} A'_f \to M \otimes_{A/tA} A'_f \to M_f \to 0$ where the first map is given by multiplication by the generator of the kernel of $A' \to A_f$ is a sequence of finite A_f-modules since $A_f \to A'_f$ is finite and is exact: it suffices to check it after localising at every point, and then by Nakayama after taking completions, but we then end of with showing the result where A/tA by some ring R' and A' by $R'[[t]]$ and the map is multiplication by t, in which case it is obvious.

[1] I chose to keep the notations of Srinivas here but they are a little bit confusing regarding the analogy of the two proofs. In the current proof, A,B and A' play respectively the roles of R, A and B in the previous proof.

One also checks that the supports of the modules are of the right codimension. Hence we have an exact sequence of functors $M^p(A/tA) \to M^p(A_f)$ with the first two terms equal and the last one given by $M \to M_f$; hence it induces 0 on K-theory. $\square$

We conclude with results relating K-theory of schemes to previously known objects of algebraic geometry. Since K_0 and K_1 are easily described for fields $k(x)$ (they are respectively $\mathbb{Z}$ and $k(x)^\times$), one may ask for a description of the map going from the K_1 to the K_0 in $E_1^{p,q}$. It is given by the following proposition:

Proposition 7. *Let X be a scheme of finite type over a field k. The image of the map d_1*

$$\bigoplus_{x \in X^{p-1}} K_1(k(x))) \to \bigoplus_{x \in X^p} K_0(k(x)) \cong \bigoplus_{x \in X^p} \mathbb{Z}$$

corresponds to codimension p cycles[2] *rationally equivalent*[3] *to 0. Hence, the cokernel of this map is isomorphic to $CH^p(X)$, the group of codimension p cycle modulo codimension p cycles rationally equivalent to 0.*

Proof. We give only a sketch. Given a point $y \in X^{p-1}$ and a point $x \in X^p$, if y specitalises to x, $k(y)$ inherits a structure of $\mathbb{Z}$-valued field with residue field $k(x)$. The statement is equivalent to the assertion that the map induced $k(y)^\times \cong K_1(k(y)) \to K_0(k(x)) \cong \mathbb{Z}$ is the corresponding valuation. Taking Y the closure of y, the functor induced by the closed immersion $Y \to X$, $M(Y) \to M(X)$ is exact and $M^i(Y)$ is sent to $M^{i+p-1}(X)$, so that we have a map of spectral sequences and so the map $k(y)^\times \cong K_1(k(y)) \to K_0(k(x)) \cong \mathbb{Z}$ can be computed on Y, i.e. we may assume X is irreducible and $p = 0$. We can also localise at x (this is flat so again functorial for the exact sequence). So we end up with $\mathcal{O}_{X,x}$ an equicharacteristic[4] discrete valuation ring, $k(x)$ its residue field and $k(y)$ its field of fractions. Taking $t \in k(y)$ of positive valuation and k_0 the prime field, we have a map $k_0[t] \to \mathcal{O}_{x,X}$ which is flat since it has no torsion and $k_0[t]$ is principal. Then to compute the image of t we are back to the original problem with $X = \mathrm{Spec}(k_0[t])$, y is the generic point, x is the closed point corresponding to $t = 0$, k_0 is $\mathbb{F}_p$ or $\mathbb{Q}$), which is not too hard. $\square$

Since K-theory coincides with G-theory for regular schemes, from proposition 4 and the previous proposition we deduce:

[2] a codimension p cycle is just a formal finite $\mathbb{Z}$-linear combination of irreducible subvarieties of codimension p

[3] given a codimension $p-1$ subvariety and a rational function on it, we get a codimension 1 cycle on this variety, which is a codimension p cycle; rational codimension p cycles are by definition sums of such cycles; two cycles are rationally equivalent iff their difference is rational.

[4] meaning the residue field and field of fraction have the same characteristic; this is true because both are k-algebras.

Proposition 8. *(Bloch's formula) Let X be a regular scheme of finite type over a field k. Denote by $\mathcal{K}_{p,X}$ the sheafififcation for the Zariski topology of the presheaf $U \to K_p(U)$. Then there are natural isomorphisms $H^p(X, \mathcal{K}_{p,X}) \cong CH^p(X)$, and we have a flasque resolution*

$$0 \to \mathcal{K}_{p,X} \to \bigoplus_{x \in X^0} (i_x)_* K_p(k(x)) \to \bigoplus_{x \in X^1} (i_x)_* K_{p-1}(k(x)) \to \cdots \to \bigoplus_{x \in X^p} (i_x)_* K_0(k(x)) \to 0$$

and isomorphisms $E_2^{p,-q}(X) \cong H^p(X, \mathcal{K}_{q,X})$ for all $p, q \geq 0$.

References

[1] Vasudevan Srinivas. *Algebraic K-theory.* Springer Science & Business Media, 2009.