DAG working group

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30/01/2026

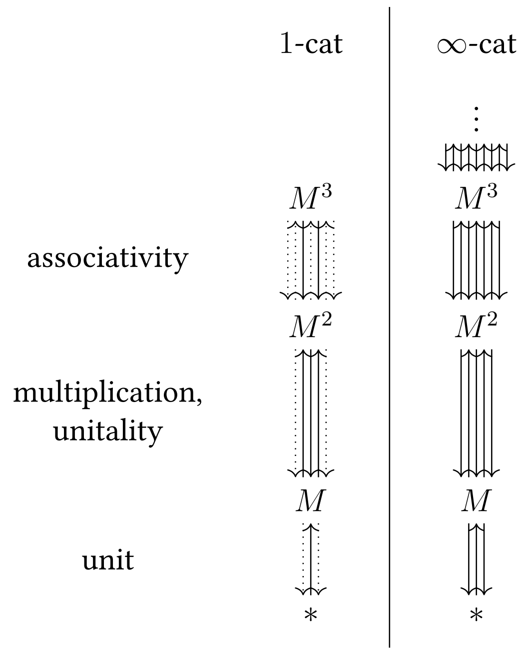
Goals for today, following [Cno27, ch. 4]:
- Monoid and group objects (non-commutative ver.)
- Looping, delooping
- The recognition principle (for loop spaces)
- Cancelled for no time: The recognition principle (for connective spectra)

1 Monoids and Groups

In our journey towards higher algebra, it is natural to seek to understand monoids, groups, and more general algebraic structures in the context of ∞-categories. Three weeks ago, in his talk on symmetric monoidal categories, Paul already introduced *commutative* monoid objects. Today we start by defining the non-commutative version. The construction is of course very similar in spirit to what he did with functors out of $\mathsf{FinSet}^{\mathsf{part}}$.

1.1 Non-commutative version

What is a monoid in an 1-category? It consists of an object M, together with a unit $* \to M$ and multiplication $M^2 \to M$, satisfying unitality and associativity constraints. These can be conveniently encoded by asking that the (left) diagram with solid arrows commutes.



We then obtain for free commutation of the dotted arrows, which don't bring any new information to the table. Therefore, in a 1-category, the monoids correspond to simplicial objects $M_\bullet$ such that for each $[n] \in \Delta$, the morphisms $e_i : M_n \to M_1$ obtained from the edge maps

$$[1] \xrightarrow{\{i,i+1\}} [n]$$

identify M_n with M_1^n. This is the so-called ***Segal condition***.

Definition 1. *Let $\mathcal{C}$ an ∞-category possessing a terminal object. A* ***monoid***[1] *in $\mathcal{C}$ is a simplicial object $M_\bullet$ of $\mathcal{C}$ verifying the Segal condition. We denote by*

$$\mathsf{Mon}(\mathcal{C}) \subseteq \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$$

the full subcategory of monoids. In the following we will almost always abuse language and write simply M both for the monoid $M_\bullet$ and the underlying object M_1.

In a 1-category, a group object is usually presented as sporting a new inversion map satisfying a bunch of new relations. Attempting to generalise this directly would lead us straight to combinatorial hell, so we won't. Instead, we reformulate "being a group" as a *property* on a monoid M. This is in fact very easy.

Definition 2. *A monoid with multiplication m is* ***grouplike****, or a* ***group****, if the shear map*

$$(\pi_1, m) : \begin{array}{c} M^2 \to M^2 \\ (x,y) \mapsto (x, xy) \end{array}$$

is an isomorphism. Let $\mathsf{Grp}(\mathcal{C})$ the full subcategory of $\mathsf{Mon}(\mathcal{C})$ consisting of grouplike monoids.

[1] Aka E_1-monoid or E_1-space.

1.2 Relation to the commutative version

Recall that Paul defined commutative monoids in $\mathcal{C}$ as the full subcategory $\mathsf{CMon}(\mathcal{C})$ of functors $\mathsf{FinSet}^{\mathsf{part}} \to \mathcal{C}$ satisfying the analoguous Segal condition. In order to define the underlying non-commutative monoid it suffices to find the appropriate inclusion $\Delta^{\mathsf{op}} \to \mathsf{FinSet}^{\mathsf{part}}$ along which to restrict.

Fix $[n] \in \Delta$. We send it to $[n-1] \in \mathsf{FinSet}$, the set with n elements (indexed from 0 to $n-1$). Note that the inner face maps act on a monoid by multiplying two adjacent elements in an $(n+1)$-tuple. We therefore assign to $d^i : [n+1] \to [n]$ (where $0 < i < n+1$) the function $[n] \to [n-1]$ defined by

$$j \mapsto \begin{cases} j & \text{if } 0 \leq j \leq i-1 \\ j-1 & \text{if } i \leq j \leq n-1. \end{cases}$$

On the other hand the outer face maps d^0, d^n acts on a monoid by forgetting the first (resp. last) element of an $n+1$-tuple. Thus we assign to d^0 the partial function sending 0 nowhere and $1 \leq j \leq n$ to $j-1$. Dually to d^n assign the partial function fixing all $0 \leq j \leq n-1$ and sending n nowhere.

Finally, a degeneracy map $s^i : [n] \to [n+1]$ simply inserts the unit at the corresponding index. We assign to it the function $[n-1] \to [n]$ defined by

$$j \mapsto \begin{cases} j & \text{if } 0 \leq j \leq i-1 \\ j+1 & \text{if } i \leq j \leq n-1. \end{cases}$$

It may be verified that these assignments verify the simplicial identities and extend to the desired fully faithful functor $\Delta^{\mathsf{op}} \to \mathsf{FinSet}^{\mathsf{part}}$.

2 Looping, delooping

2.1 Looping

In the homotopy I course, we have seen that the loop object ΩX of a pointed topological space admits a group structure in hTop. Recall the definition,

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Path}(X) \arrow[d] \\ * \arrow[r, "x"'] & X. \end{tikzcd}$$

As the map $\mathrm{Path}(X) \to X$ is a fibration, this strict pullback is in fact a homotopy pullback; and as $\mathrm{Path}(X) \simeq *$ it may be better understood as the homotopy pullback $\Omega X = * \times_X *$.

In a general ∞-category $\mathcal{C}$, this is how we define the loop object of a pointed X. We will now endow it with a (natural!) group structure, which in the specific case of $\mathcal{C} = \mathsf{An}$ retrieves the one we know and love upon passing to the homotopy category. In order to perform this feat, we seek a simplicial object extending $\Omega X \to *$. There is a natural candidate one, the Čech nerve of the map $x : * \to X$!

Recall that this is a simplicial object $\check{C}(x)_\bullet$ whose n-simplices are given by

$$\check{C}(x)_n = * \times_X * \times_X \cdots \times_X * = *_X^{n+1}.$$

Some abstract nonsense (see [Cno27, lem. 4.1.9] for details) inform us that the edge maps $e_i : \check{C}(x)_n \to \check{C}(x)_1 = \Omega X$ do indeed identify $\check{C}(x)_n$ with Ω_X^n.

Remark 3. The Čech nerve is functorial, and can be written as the composite

$$\check{C}_\bullet : \mathsf{Fun}^*([1],\mathcal{C}) \simeq \mathsf{Fun}([1]^{\mathrm{op}},\mathcal{C}) \xrightarrow{j_*} \mathsf{Fun}(\Delta_+^{\mathrm{op}},\mathcal{C}) \xrightarrow{i^*} \mathsf{Fun}(\Delta^{\mathrm{op}},\mathcal{C}),$$

where j_* is the right Kan extension along the inclusion $j : [1]^{\mathrm{op}} \to \Delta_+^{\mathrm{op}}$ sending 0 to $[-1]$ and 1 to $[0]$, and i^* is the restriction along the inclusion $i : \Delta^{\mathrm{op}} \to \Delta_+^{\mathrm{op}}$. See [Cno27, def. 4.1.6 and disc. 4.1.7] for details.

Let's check that this is sensible by looking closer at the case of Top with the Quillen model structure. By replacing the maps $* \to X$ with the fibration $\mathrm{Path}(X) \to X$, we can compute explicitly

$$\check{C}(x)_n = \mathrm{Path}(X)_X^{n+1}.$$

I like to view this as the space of $(n+1)$ tendrils extending from x and rejoigning at some other unspecified point.

On figure 1, we see that the "abstract" monoid structure on $\check{C}(x)$ corresponds exactly to the "concrete" monoid structure on ΩX. This is reassuring. Now examples are good and all, but we still have to prove something.

Proposition 4. *Let $x : * \to X$ a pointed object in an ∞-category $\mathcal{C}$ with the appropriate pullbacks. Then $\check{C}(x)$ is a grouplike monoid with underlying object ΩX.*

We will write equally ΩX for the object and the group object. When the two need to be distinguished, the object shall be denoted by $(\Omega X)_1$.

Proof. Because we have admitted that $\check{C}(x)_n \simeq (\Omega X)^n$ through $(e_0, \dots, e_{n-1})$, it remains only to check that the shear map $(\pi_1, m) : \check{C}(x)_2 \to \check{C}(x)_2$ is an isomorphism. Moreover it suffices to check the statement in the case of animae by an application of Yoneda lemma. Indeed, for any object Y of $\mathcal{C}$,

$$\mathrm{Map}(Y, \Omega X) = \Omega \mathrm{Map}(Y, X)$$

and the shear map for X is sent to the shear map for $\mathrm{Map}(Y,X)$. Furthermore, whether a map is an isomorphism can be tested at the level of homotopy categories. But we already know that (the weak homotopy type of) the loop space of a pointed topological space is a group object! $\square$

2.2 Delooping

Much like how it is false in general that $\Sigma\,\Omega X \simeq X$, we cannot always recover X from ΩX. However, we will see shortly that in the case of animae, we actually remember the whole connected component of the chosen point x. The best approximation of an inverse operation to Ω is the ***classifying space*** B.

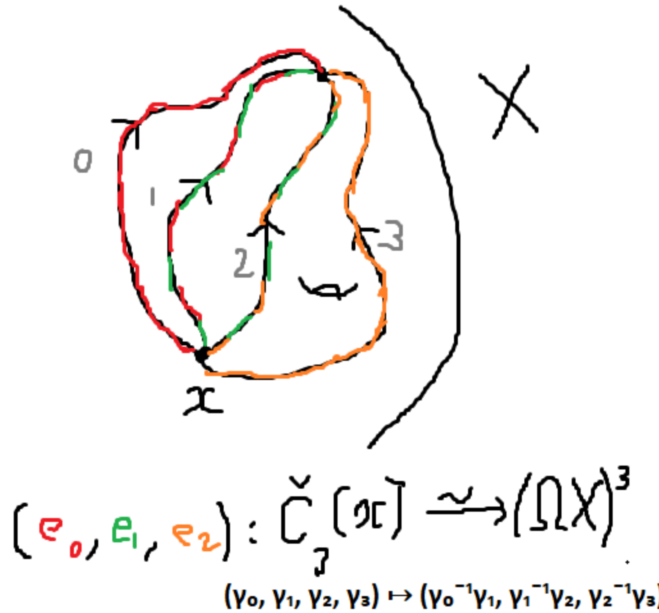
Definition 5. *Let $\mathcal{C}$ an ∞-category with the appropriate geometric realizations, and $M_\bullet$ a monoid. Its **classifying space** BM is the colimit*

$$BM = |M| = \operatorname*{colim}_{\Delta^{\mathrm{op}}} M_\bullet \in \mathcal{C}.$$

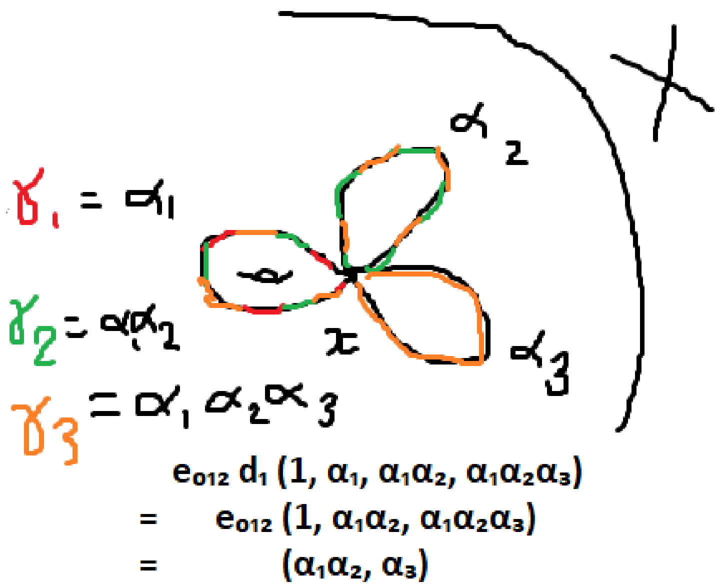
This object is naturally pointed by the map $* = M_0 \to |M| = BM$, defining a functor

$$B \colon \mathsf{Mon}(\mathcal{C}) \to \mathcal{C}_*.$$

(a) An element of $\check{C}(x)_3$.



(b) The map $e_{012} \coloneqq (e_0, e_1, e_2)$.



(c) The map $(\Omega X)^3 \to (\Omega X)^2$ induced by d^1.

Figure 1: The concrete group structure on ΩX obtained from the abstract group structure on $\check{C}(x)$.

Remark 6. By the pointwise formula for Kan extensions, the functor B may be expressed as the composite

$$\mathsf{Mon}(\mathcal{C}) \subseteq \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \xrightarrow{i_!} \mathsf{Fun}(\Delta_+^{\mathrm{op}}, \mathcal{C}) \xrightarrow{j^*} \mathsf{Fun}([1]^{\mathrm{op}}, \mathcal{C}),$$

where $i_!$ is the left Kan extension along the inclusion $i : \Delta \to \Delta_+$ and j^* the restriction along the same inclusion as in remark 3.

Proposition 7. *Let $\mathcal{C}$ an ∞-category with finite limits and geometric realizations. The loop space and classifying space functors define an adjunction*

$$B\colon \mathsf{Mon}(\mathcal{C}) \dashv \mathcal{C}_* : \Omega.$$

Proof. We have a more general adjunction

$$\begin{tikzcd} \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\Delta_+^{\mathrm{op}}, \mathcal{C}) \arrow[l, shift left, "i^*"] \arrow[r, shift left, "j^*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}([1]^{\mathrm{op}}, \mathcal{C}) \arrow[l, shift left, "j_*"] \end{tikzcd}$$

of which B and Ω are the restriction to full subcategories. □

Remark 8. Remark that over animae (or any ∞-topos), B preserves products. In particular, it preserves monoid and group objects. If we believe in the heuristic that an abelian group is a group object in group objects, this would mean that whenever A is an abelian group object, so is BA. I wanted to prove this but there was no time.

3 The recognition principle for loop spaces

What happens to ΩBM if M was already a group? Under which conditions does X equal $B\Omega X$? All of those deeply existential questions and more are answered by the following

Theorem 9 (Recognition principle). *Let M a monoid in* An *and let X a pointed anima. Then*

(a) The unit map $M \to \Omega BM$ is an isomorphism if and only if M is grouplike.

(b) The pointed anima BM is always connected, and in fact

(c) The counit $B\Omega X \to X$ is the inclusion of the connected component of the basepoint.

In particular, the adjunction $B \dashv \Omega$ restricts to an equivalence $\mathsf{Grp}(\mathsf{An}) \to \mathsf{An}_*^{\geq 1}$ *between animated groups and pointed connected animae.*

Slogan in *flexeur* language: any grouplike E_1-space is equivalent to a loop space.

Remark 10. The same is true in a presheaf topos because all these constructions are taken pointwise. In fact it holds in any ∞-topos, because left-exact localisation commutes with both colimits (hence B) and finite limits (hence Ω).

Proof. [Cno27, p. 131] We first prove claim (b). The $\pi_0 : \mathsf{An} \to \mathsf{Set}$ is left adjoint to inclusion of sets as discrete spaces, so commutes with colimits. Hence

$$\pi_0(BM) = \operatorname*{colim}_{\Delta^{\mathrm{op}}} \pi_0(M_\bullet) = \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \left[\cdots \dashrightarrow \pi_0 M^2 \;\substack{\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow}\; \pi_0 M \;\substack{\rightarrow\\\leftarrow\\\rightarrow}\; * \right] = *$$

so that BM is connected.

Assume (a) for now and prove (c). As $B\Omega X$ is connected, the counit lands in the connected component X°, and we have to show $B\Omega X \to X^\circ$ is an isomorphism. Because both sides are connected and homotopy groups form a conservative family of functors, it suffices to check that

$$\Omega X \simeq \Omega B \Omega X \to \Omega X^\circ$$

is an isomorphism. This is by definition of Ω.

It remains to show the most interesting claim (a). The forward direction follows from proposition 4. For the converse direction, it suffices to prove that $M_1 \to (\Omega BM)_1$ is an isomorphism as it will then induce an isomorphism at all levels. This map comes from the commutative square

$$\begin{tikzcd} M_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & M_0 \simeq * \arrow[d] \\ M_0 \simeq * \arrow[r] & BM, \end{tikzcd}$$

that we must check is cartesian.

For this, we shall use the following

Theorem 11 (Universality of colimits [HTT, 6.1.3.9])**.** *Let I a small ∞-categories, $\bar{F}, \bar{G} : I^{\triangleright} \to \mathsf{An}$ cocones over functors F, G, and $\bar{\alpha} : \bar{F} \to \bar{G}$ a natural transformation. Assume that the restriction $\alpha : F \to G$ is cartesian, in the sense that for every $i \to j$ in I the square*

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\alpha_i"] & F(j) \arrow[d, "\alpha_j"] \\ G(i) \arrow[r] & G(j) \end{tikzcd}$$

is cartesian. Assume further $\bar{G}$ to be a colimit. Then $\bar{F}$ is a colimit if and only if $\bar{\alpha}$ is cartesian, which is to say all the squares

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\alpha_i"] & \bar{F}(\infty) \arrow[d, "\bar{\alpha}_\infty"] \\ G(i) \arrow[r] & \bar{G}(\infty) \end{tikzcd}$$

are cartesian also.

We admit this and carry on with the proof of the recognition principle.

Let $I = \Delta^{\mathrm{op}}$ so that $I^{\triangleright} = \Delta_+^{\mathrm{op}}$, as well as $G = M$ and $\bar{G}(\infty) = BM$. Set $\bar{F}[n] = \bar{G}[n+1]$, obtained from $\bar{G}$ by discarding the purple arrows below:

$$\begin{tikzcd}
 & \bar{F}[1] & & \bar{F}[0] & & \bar{F}[-1] & & \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] \arrow[r, shift right=3] & M^2 \arrow[rr, shift left=2, "d^0"] \arrow[rr, shift left=0.7] \arrow[rr, shift right=0.7] \arrow[rr, shift right=2, "d^2"'] & & M \arrow[rr, shift left, "d^0"] \arrow[rr, shift right, "d^1"'] & & * \arrow[rr] & & BM \\
 & \bar{G}[2] & & \bar{G}[1] & & \bar{G}[0] & & \bar{G}[-1]
\end{tikzcd}$$

Define also $\bar{\alpha}_n = d_{n+1}$. If we knew that α was cartesian and $\bar{F}$ a colimit, then universality would give us exactly the required cartesian square for $i = 0$. That $\bar{F}$ is a colimit follows from another very general lemma, whose statement and proof we defer until later. Now for cartesianity of α, we prove this by cases.

Case 1 – The map $d^1 : [1] \to [0]$. In this case, the square reduces to

$$\begin{tikzcd}
M^2 \arrow[r, "m"] \arrow[d, "\pi_0"'] & M \arrow[d] \\
M \arrow[r] & *
\end{tikzcd}$$

which is a pullback precisely because M is grouplike.

Case 2 – The map $d^0 : [n+1] \to [n]$. The square reduces to

$$\begin{tikzcd}
M^{n+2} \arrow[r, "(m_0 \dots m_{n+1}) \mapsto (m_1 \dots m_{n+1})"] \arrow[d, "(m_0 \dots m_{n+1}) \mapsto (m_0 \dots m_n)"'] & M^{n+1} \arrow[d, "(m_0 \dots m_n) \mapsto (m_0 \dots m_{n-1})"] \\
M^{n+1} \arrow[r, "(m_0 \dots m_n) \mapsto (m_1 \dots m_n)"'] & M^n
\end{tikzcd}$$

which is always a pullback.

Case 3 – The map $d^i : [n+1] \to [n]$ for $0 < i < n+1$. The square reduces to

$$\begin{tikzcd}
M^{n+2} \arrow[r, "(m_0 \dots m_{n+1}) \mapsto (m_0 \dots, m_{i-1} m_i, \dots m_{n+1})"] \arrow[d, "(m_0 \dots m_{n+1}) \mapsto (m_0 \dots m_n)"'] & M^{n+1} \arrow[d, "(m_0 \dots m_n) \mapsto (m_0 \dots m_{n-1})"] \\
M^{n+1} \arrow[r, "(m_0 \dots m_n) \mapsto (m_0 \dots, m_{i-1} m_i, \dots m_n)"'] & M^n
\end{tikzcd}$$

which is also always a pullback.

Case 4 – The map $d^{n+1} : [n+1] \to [n]$. The square reduces to

$$\begin{tikzcd}
M^{n+2} \arrow[r, "(m_0 \dots m_{n+1}) \mapsto (m_0 \dots m_n m_{n+1})"] \arrow[d, "(m_0 \dots m_{n+1}) \mapsto (m_0 \dots m_n)"'] & M^{n+1} \arrow[d, "(m_0 \dots m_n) \mapsto (m_0 \dots m_{n-1})"] \\
M^{n+1} \arrow[r, "(m_0 \dots m_n) \mapsto (m_0 \dots m_{n-1})"'] & M^n
\end{tikzcd}$$

which is the square of Case 1 – multiplied by M^n, hence a pullback.

Case 5 – The map $s^i : [n] \to [n+1]$. As s^i is a section of d^i, by the pasting law and the previous steps, the required square is a pullback.

Case 6 – A general map in Δ_+^{op}. It factors as a product of faces and degeneracies. By the pasting law, the required square is a pullback.

This finishes the proof. ☐

As promised, here is the lemma for checking that $\bar{F}$ is a colimit. Introduce first the category of extra degeneracies, Δ_+^{deg}, as the wide subcategory of Δ whose morphisms are the maps $\varphi : [n] \to [m]$ satisfying $\varphi(n) = m$. This includes all the codegeneracies, and all the coface maps save for $d_{n+1} : [n] \to [n+1]$. As a result, a functor out of $(\Delta_+^{\mathrm{deg}})^{\mathrm{op}}$ looks like a diagram

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_{1+1} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_{0+1} \;\substack{\longrightarrow\\ \longleftarrow}\; X_{-1+1}$$

in which we understand the red arrows as "extra degeneracies". The index notation was not chosen randomly; there is a faithful functor $(+1) : \Delta_+ \to \Delta_+^{\mathrm{deg}}$ sending $[n]$ to $[n+1]$.

Lemma 12 (Extra degeneracies argument)**.** *Let $\mathcal{C}$ and ∞-category and $\bar{X} : \Delta_+^{\mathrm{op}} \to \mathcal{C}$ an augmented simplicial object. Suppose that $\bar{X}$ has extra degeneracies, in the sense that $\bar{X} = \bar{Y}|_{\Delta_+}$ for some $\bar{Y} : (\Delta_+^{\mathrm{deg}})^{\mathrm{op}} \to \mathcal{C}$. Then $\bar{X}$ is a colimit cocone.*

Proof. The composite inclusion $\Delta \subseteq \Delta_+ \hookrightarrow \Delta_+^{\mathrm{deg}}$ is left adjoint to the inclusion $\Delta_+^{\mathrm{deg}} \subseteq \Delta$. Indeed for any natural numbers n, m any ordered map $\varphi : [n] \to [m]$ extends uniquely to $[n+1] \to [m]$ if we impose the condition that $\varphi(n+1) = m$. By [Cno27, cor. 19.5.6], this implies that $(+1)|_{\Delta}$ is initial, and hence that

$$(+1)^{\mathrm{op}} : \Delta^{\mathrm{op}} \to (\Delta_+^{\mathrm{deg}})^{\mathrm{op}}$$

is final. In other words

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} X_\bullet = \operatorname*{colim}_{\Delta^{\mathrm{op}}} Y_{\bullet+1} = \operatorname*{colim}_{(\Delta_+^{\mathrm{deg}})^{\mathrm{op}}} Y_\bullet = Y_0 = X_{-1}$$

because $[0]$ is terminal in $(\Delta_+^{\mathrm{deg}})^{\mathrm{op}}$. ☐

References

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