MODEL CATEGORIES AND SIMPLICIAL SETS

Part 1) A quick lecture on simplicial sets

Motivation

Defining an abstract notion of "space" which departs from the usual topological definition has far reaching applications. Take for instance the historical definition of the cohomology of a group G. First you create the Eilenberg-Maclane space $BG = K(G, 1)$ which classifies the group G (ie. $\pi_1(|BG|) = G$ and all other homotopie groups vanish) and then take the singular cohomology of BG, defining $H^n(G, \mathbf{Z}) = H^n_{sing}(BG, \mathbf{Z})$. From afar, this is strange, we are defining an algebraic proprety of an algebraic object via topological structures. This motivates us to look for a notion of space that is abstracted away into more general structures.

The bare bones approach to defining a space would be as a collection of sets

$$X_0 \qquad X_1 \qquad X_2 \qquad \cdots$$

where X_n is the set of n-simplices of X. If we add the condition that any face of an n-simplex is also in our collection then we would have defined an *abstract simplicial complex.*

However as is, this is too little structure, and we can see this categorically by taking products or collapsing subspaces: $\Delta^1 \times \Delta^1 \cong \Delta^3$ and $\Delta^1/\partial\Delta^1 \cong *$. This is not what we'd expect, in other words these operations do not commute with their *geometric realization.*

The solution is to add more structure by identifying faces using *face maps*:

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left] \arrow[l, shift right] & X_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \end{tikzcd}$$

This fixes the issues described above, and is the idea behind *semi-simplicial sets.* However one last issue is that we'd sometimes like to manipulate degenerate simplices, for instance we would like to glue the border of an n-simplex to a single point, similar to how in CW-complexes we glue the border of the n-ball to a point to obtain the n-sphere. However as defined, we are only able to easily glue along $(n-1)$-simplices. To fix this we add degenerate morphisms (going in the opposite direction in the diagram below):

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] & X_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \end{tikzcd}$$

which will allow us, for instance, to view an 0-simplex (ie. a point) as a degenerate n-simplex. This is now the idea for *simplicial sets.*

Simplicial sets

All of this data can be packaged amazingly well as follows:

Definition 1. Let Δ be the category which has objets $\underline{n} = \{0 < ... < n\}$ and morphisms are order-preserving functions. The category of **simplicial sets** sSet is $\mathrm{PSh}(\Delta) = \widehat{\Delta} = [\Delta^{\mathrm{op}}, \mathrm{Set}]$.

This highly compact definition is equivalent to our previous considerations where for $X \in \mathrm{sSet}$ we write $X_n = X(\underline{n})$. The correspondence is detailed using the following definitions:

Definition 2. We denote $d^i : \underline{n-1} \to \underline{n}$ as the unique injection which skips i.

Intuitively, the precomposition $d_i = (d^i)^* : X_n \to X_{n-1}$ selects the i-th face of an n-simplex.

Definition 3. We denote $s^i : [n+1] \to [n]$ the unique surjection such that $s^i(i) = s^i(i+1)$.

Now $s_i = (s^i)^*$ takes an n-simplex and views it as an $(n+1)$-simplex where we've squished the i-th and $(i+1)$-st vertex together.

Proposition 4. *The d^i and s^i are subject to the cosimplicial relations :*

1. $d^j d^i = d^i d^{j-1}$ *if* $i < j$.
2. $s^j s^i = s^i s^{j+1}$ *if* $i \leqslant j$.
3. $s^j d^i = d^i s^{j-1}$ *if* $i < j$.
4. $s^j d^i = \mathrm{id}$ *if* $i = j$ *or* $j+1$.
5. $s^j d^i = d^{i-1} s^j$ *if* $i > j+1$.

One can obtain the simplicial relations by dualizing (ie. flipping the order).

Proposition 5. *The d_i and s_i are subject to the simplicial relations :*

1. $d_i d_j = d_{j-1} d_i$ *if* $i < j$.
2. $s_i s_j = s_{j+1} s_i$ *if* $i \leqslant j$.
3. $s_j d_i = d_i s_{j-1}$ *if* $i < j$.
4. $s_j d_j = s_j d_{j+1} = \mathrm{id}$ *if* $i = j$.
5. $s_j d_i = d_{i-1} s_j$ *if* $i > j+1$.

From a family of morphisms satisfying these relations we can reconstruct a simplicial set.

Definition 6. Let $\mathbb{A}^n = \{(x_0, ..., x_n) \in \mathbf{R}^{n+1} | \sum x_i = 1, x_i \geqslant 0\}$ be the topological n-simplex.

Definition 7. Let X be a topological space. The **singular simplex** of X is defined as $(SX)_n = \mathcal{C}(\mathbb{A}^n, X)$. This defines a functor $S : \mathrm{Top} \to \mathrm{sSet}$.

This functor has a left adjoint $|-| : \mathrm{sSet} \to \mathrm{Top}$. For $X \in \mathrm{sSet}$, we have

$$|X| = \coprod_{n \in \mathbf{N}} X_n \times \Delta^n_{top} / \sim$$

where $\sim$ is generated via s_i and d_i.

Definition 8. The n-standard simplicial set is $\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$.

By Yoneda, a morphism $\Delta^n \to X$ corresponds to an n-simplex of X.

Remark 9. We have that $|\Delta^n| = \mathbb{A}^n$.

Recall the density theorem : Every presheaf is a colimit of representables.

$$P \cong \operatorname*{colim}_{y(X) \to P} y(X)$$

In our context this means that a simplicial set is the gluing of all of its simplices.

Proposition 10. *For any simplicial set X we have*

$$X \cong \operatorname*{colim}_{\Delta^n \to X} \Delta^n$$

Definition 11. An n-simplex $x \in X_n$ is **nondegenerate** if it is not in the image of a degeneracy map (ie $x \notin s_i(X_{n-1})\ \forall i$) We write $\widetilde{X_n}$ for the set of nondegenerate n-simplicies.

Exercise 12 (Zilber-Eilenberg)**.** For any n simplex $x \in X_n$, there is a unique non-degenerate simplex $y \in X_m$ and a unique surjection $f : \underline{m} \to \underline{n}$ such that $f^*(y) = x$.

The nondegenerate simplices contain all the "visual information" of a simplcial set, for instance δ^n has many simplices, but exactly one nondegenerate one. To see what I mean by "visual information" you can observe the following exercise (which follows from the Zilber-Eilenberg lemma).

Exercise 13. There is a continuous bijection $\coprod_{n \geqslant 0} \widetilde{X_n} \times (\Delta^n_{top} \setminus \partial\Delta^n_{top}) \to |X|$ (which is generally not a homeomorphism).

Other constructions which are important but we will not have time to elucidate:
• $\partial\Delta^n = \mathrm{coeq}(\coprod \Delta^{n-2} \to \coprod \Delta^{n-1})$. The maps here are precisely every possible injection of the form $\underline{n-2} \to \underline{n-1}$. And the k simplices of this simplicial set are morphisms $[k] \to [n]$ which are not surjective.
• $\mathrm{Map}(X,Y) \in \mathrm{sSet}$ is defined as $\mathrm{Map}(X,Y)_n = \mathrm{Hom}(\Delta^n \times X, Y)$.
• $S^n = \Delta^n / \partial\Delta^n$.

Two maps $f, g : S^n \to X$ (which correspond to n-simplices which are sent to the same 0-simplex x via face maps) are homotopic if there exists a morphism $h : \Delta^n \to \mathrm{Map}(\Delta^1, X)$ such that

(1) $\Delta^n \to \mathrm{Map}(\Delta^1, X) \to X \times X$ yields (f, g),

(2) $h|_{\partial\Delta^n}$ is constant equal to x.

Then we can also define the (higher) homotopy groups : $\pi_n(X,x) = \{f : S^n \to X | f(\partial\Delta^n) = x.\}$

1 Model categories

1.1 Localization

Let $\mathscr{C}$ be a category, $\mathcal{W}$ a class of morphisms called **weak-equivalences** which satisfy the two out of three property (if f, g are composable morphisms and two out of three of f, g and gf are in $\mathcal{W}$, then so is the third). We define $\mathscr{C}[\mathcal{W}^{-1}]$ as the category with the same objects as $\mathscr{C}$ but formally inverting all the morphisms in $\mathcal{W}$. It has a universal property : If $\mathscr{C} \to \mathscr{D}$ send all the morphisms in $\mathcal{W}$ to isomorphisms, then there is a unique morphism $\mathscr{C}[\mathcal{W}^{-1}] \to \mathscr{D}$ making the diagram commute.

Definition 14. We say a class of arrows $\mathcal{A}$ has the two out of three property if for all composable arrows f, g, if two of the following $f, gf \circ g$ are in $\mathcal{A}$ then all are.

Definition 15. Given $f : A \to B, g : A' \to B'$ two morphisms in $\mathscr{C}$, we say that f is **orthogonal** to g and write $f \perp g$ (the order is important) if for all $\alpha : A \to A'$ and $\beta : B \to B'$, there exists $h : B \to A'$ making the two triangles commute

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & B' \end{tikzcd}$$

We write $A^\perp = \{g | f \perp g \ \forall f \in A\}$ and ${}^\perp A = \{g | g \perp f \ \forall f \in A\}$.

Definition 16. We say that (A, B) is a **weak factorisation system** if
• $A = {}^\perp B$ and $A^\perp = B$.
• All $f \in \mathscr{C}$ factorizes functorially as a composition $p \circ i$ with $i \in A$ and $p \in B$.

Definition 17. A **model category** M is a category equipped with classes of morphisms $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ such that
• M is complete and cocomplete.
• $\mathcal{W}$ satisfies the two out of three property and contains all isomorphisms.
• $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are stable under retracts. Just be careful that retracts here are defined as follows (ie. retracts in the category of arrows of M). A morphism $f : A \to B$ is a retract to $g : X \to Y$ if there exist

morphisms making the following diagram commute :

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r] & B
\end{tikzcd}$$

- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorisation systems.

Here are the most important examples of model categories, one should keep in mind that model structure are not *intrinsic* to the homotopy of a category, but are instead additional structure. For instance the model structures given below (the *Quillen model structures*) are probably their most important model structure but they certainly are not the only ones. (the Hurewicz model is another such one for Top)

By the way, by Top we actually mean CGTop, the space of compactly generated topological spaces. This is done for technical reasons.

	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{F}$
Top	f such that f_* iso on all π_n	retracts of relative CW-complexes generated by $S^n \to D^n$	Serre fibration
sSet	same	monomorphisms	Kan fibrations
Ch_R (projective)	quasi-iso: f_* iso on all H_n	monos degree-wise with projective cokernels	degree-wise surjections
Ch_R (injective)	same	monos degree-wise	degree-wise surjections with degree-wise injectives

Table 1: Examples

Definition 18. An object $X \in M$ is **cofibrant** if $\varnothing \to X$ is a cofibration. It is **fibrant** if $X \to *$ is a fibration.

Notice that every object X has a morphism $\varnothing \to X$ which can be split $\varnothing \to \widetilde{X} \to X$ with $\widetilde{X} \to X$ a weak equivalence and $\widetilde{X}$ a cofibrant object. Since this factorization is, by definition, functorial, we can produce a functor $Q : M \to M$ such that $\varnothing \to Q(X) \to X$ with $Q(X) \to X$ is a weak equivalence and $Q(X)$ is cofibrant. We call this a **cofibrant replacement** (not unique). Likewise, there is a **fibrant replacement** denoted R.

	Cofibrant objects	Fibrant objects
Top	retracts of CW-complexes	every object
sSet	every object	Kan fibrant objects
Ch_R (projective)	degree-wise projective	every object
Ch_R (injective)	every object	degree-wise injectives

Table 2: Cofibrant and fibrant objects

It can then be seen that the process of projective resolution in chain complexes arises as a cofibrant replacement in the projective model structure.

Definition 19 (smallness)**.** We say that an object C is small (we also call it categorically compact or finitely generated) if any arrow of C into a filtered colimit *factors*:

$$X \to \operatorname*{colim}_D X_i$$

$$X \to X_i \to \operatorname*{colim}_D X_i$$

Remark 20. Despite the name, the "compact" object of the usual Top are not compact sets, but *finite discrete sets.* In CGTop they do coincide with compact spaces, which is part of why we use them instead.

Proposition 21 (Small object argument)**.** *Idea : Let $I = \{i_\alpha : A_\alpha \to B\}$ where A_α is small . We say that X is small if $X \to \operatorname{colim}_D X_i$ factors through an X_i where D is filtered. Then $({}^\perp(I^\perp), I^\perp)$ is a weak factorisation system.*

A continuation of model category would then speak of :
- Cofibrantly generated models and how to use the small object argument to create them. - Quillen adjunctions (the natural morphism for model categories)
- Quillen equivalences. - Derivations of functors.
- Homotopy colimits.