Some basic category theory

Jiangfan Y.

Contents

1 Model Categories

1.1 Definition of model categories

To quote [Lur09]: *Quillen's theory of model categories provides a useful tool for studying specific examples of ∞-categories, including the theory of ∞-categories itself.*

Definition 1.1. Let C be a category, with three classes of maps:

$$\begin{aligned} &(\mathsf{W}) \text{ Weak equivalences } \bullet \xrightarrow{\sim} \bullet; \\ &(\mathsf{C}) \text{ Cofibrations } \bullet \hookrightarrow \bullet; \\ &(\mathsf{F}) \text{ Fibrations } \bullet \twoheadrightarrow \bullet. \end{aligned}$$

So that each class of maps is closed under composition of maps and $\mathrm{id}_X, \forall X \in C$ locates in all three classes.
We call maps in $(\mathsf{W}) \cap (\mathsf{C})$ **acyclic cofibrations**, whose arrows are $\overset{\sim}{\hookrightarrow}$, and maps in $(\mathsf{W}) \cap (\mathsf{F})$ **acyclic fibrations**, with arrows $\overset{\sim}{\twoheadrightarrow}$.
We say C is a **model category** if the following 5 axioms hold:
MC1 C is complete and cocomplete.
MC2 For $f, g \in \mathrm{Mor}(C)$ and gf defined, then any two of f, g, gf are weak equivalences, then the third one is.
MC3 If f is a retract of g in $\mathrm{Mor}(C)$, then whether g is in (W) or (C) or (F), so is f.
MC4 Consider the following commutative diagram (where i is a cofibration and p is a fibration):

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

Then whenever (a) i is an acyclic cofibration or (b) p is an acyclic fibration, there exists a lifting $h : B \to X$ so that $f = hi$ and $g = ph$.
MC5 For any map $f : X \to Y$, there exists two factorization of f, (a) $f = pi$, where p is a fibration and i is an acyclic cofibration; (b) $f = pi$, where p is an acyclic fibration and i is a cofibration.

Since an initial object in the colimit over empty diagram and a terminal object is the limit over empty diagram, by **MC1**, we know that they exist in model category C, denoted by $\varnothing$ and $*$ respectively.

Definition 1.2. An object $X \in C$ is called a **cofibrant** if $\varnothing \to X$ is a cofibration, it is called a **fibrant** if $X \to *$ is a fibration. Denote by:
C_{c}: full subcategory of C with objects are cofibrants.
C_{f}: full subcategory of C with objects are fibrants.
C_{cf}: full subcategory of C with objects are both cofibrants and fibrants.

Remark 1.3. Use **MC5** on $X \to *$ and $\emptyset \to X$, we get RX fibrant and QX cofibrant with $i_X : X \overset{\sim}{\hookrightarrow} RX$, $p_X : QX \overset{\sim}{\twoheadrightarrow} X$.

Proposition 1.4. *(i) The class of cofibrations in C is stable under cobase change, the same for acyclic cofibrations.*
(ii) The class of fibrations in C is stable under base change, the same for acyclic fibrations.

Proof. Consider the pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & D \end{array}$$

where i is a cofibration map and f is the cobase change map. We would like to prove that j is a cofibration. This uses the following criterion for (co)fibrations in model category.
We say $i : A \to B$ have left lifting property (*LLP*) to $p : X \to Y$ if for any commutative diagram as 1, there is a lifting $B \to X$. In such case, p is said to have right liftng property (*RLP*) to i.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array} \tag{1}$$

Lemma 1.5. *(i) The cofibrations in C are those who have LLP with respect to acyclic fibrations, and the acyclic cofibrations in C are those who have LLP with respect to fibrations.*
(ii) The fibrations in C are those who have RLP with respect to acyclic cofibrations, and the acyclic fibrations in C are those who have RLP with respect to cofibrations.

Proof of Lemma 1.5. One side is clear, now assume $i : A \to B$ has LLP for any acyclic fibration. We can factorize i as $A \overset{i'}{\hookrightarrow} B' \overset{p'}{\twoheadrightarrow} B$ so that p' is acyclic. Use LLP for $B' \overset{\sim}{\twoheadrightarrow} B$, there is a lifting $j : B \to B'$ so that $i' = ji$, $p'j = \mathrm{id}_B$. We have the following diagram that makes i a retract of i':

$$\begin{array}{ccccc} A & \overset{\mathrm{id}_A}{=\!=\!=} & A & \overset{\mathrm{id}_A}{=\!=\!=} & A \\ {\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{j} & B' & \overset{p'}{\twoheadrightarrow} & B \end{array}$$

Use the axiom **MC3** we conclude that i is a cofibration as i' is. The rest cases can be similarly proved. ∎

Now by Lemma 1.5, we only need to prove that j has *LLP* to acyclic fibrations. We consider:

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & \sim\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & D & \xrightarrow[b]{} & L \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{f} & C & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle l_1}\nearrow & {\scriptstyle l_2}\nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & D & \xrightarrow[b]{} & Y \end{array}$$

where the left diagram is commutative and $p : X \to Y$ is an acyclic fibration. Since i is cofibration and p is acyclic fibration, by **MC4** we have a lifting $l_1 : B \to X$ so that $l_1 i = af$. By universal property of pushout, we have a map $l_2 : D \to X$ so that $l_2 g = l_1$. Now l_2 is in fact a lifting in the diagram

$$\begin{array}{ccc} C & \xrightarrow{a} & X \\ {\scriptstyle j}\downarrow & \overset{l_2}{\nearrow} & \downarrow{\scriptstyle p} \\ D & \xrightarrow[b]{} & Y \end{array}$$

This shows j is LLP to p and hence j is a cofibration. When i is an acyclic cofibration, we can loose the condition on p that requiring it to be just a fibration. In this case, we can see j will be acyclic. The proof of (ii) is dual. □

1.2 Homotopy category of a model category

We fix C a model category.

Definition 1.6. Let $A \in C$ be an object, a **cylinder object** of A is an object $A \wedge I$ so that $\mathrm{id}_A + \mathrm{id}_A : A \coprod A \to A$ factor through $A \wedge I$ as $A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$ where the factorized map $A \wedge I \to A$ is a weak equivalence. Denote by i_0, i_1 the composition $A \to A \coprod A \xrightarrow{i} A \wedge I$ corresponding to $A \to A \coprod A$ by first/second factor respectively.
A cylinder object $A \wedge I$ is called:
(a) *good* if $A \coprod A \to A \wedge I$ is a cofibration.
(b) *very good* if $A \wedge I$ is good and $A \wedge I \to A$ is an acyclic fibration.

We have the dual notion of cylinder object:

Definition 1.7. Let $X \in C$ be an object, a **path object** of X is an object X^I so that $(\mathrm{id}_X, \mathrm{id}_X) : X \to X \times X$ factor through X^I and the factorized map $X \to X^I$ is a weak equivalence. A path object X^I is called:
(a) *good* if $X^I \to X \times X$ is a fibration.
(b) *very good* if $A \wedge I$ is good and $X \to X^I$ is an acyclic cofibration.

We have the notion of homotopy (of maps).

Definition 1.8. Let $f, g : A \to B$ be two maps, we say that f is **left homotopic to** g if $f + g : A \coprod A \to B$ factors some cylinder object $A \wedge I$ by $H : A \wedge I \to B$ that satisfies $Hi_0 = f$, $Hi_1 = g$. Denote by $f \overset{l}{\sim} g$, and H is called the *left homotopy* from f to g (via $A \wedge I$). H is called good or very good if $A \wedge I$ is.
The notion of f right homotopic to g is dually defined. Namely if $f, g : A \to B$ two maps, f is **right homotopic to** g if $(f, g) : A \to B \times B$ factors through some path object B^I by *right homotopy* $H : A \to B^I$. Denote by $f \overset{r}{\sim} g$. The goodness of H is again dependent on B^I.

We note that if $f \overset{l}{\sim} g$, a good left homotopy always exists between f and g as we can apply **MC5** to $i : A \coprod A \to A \wedge I$. Similar for the case when $f \overset{r}{\sim} g$.

Lemma 1.9. *Let $f, g : A \to B$ be two maps, then:*
(i) If A is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$.
(ii) If B is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$.

Lemma 1.10. *If A is cofibrant and $A \wedge I$ a good cylinder object of A, then $i_0, i_1 : A \to A \wedge I$ are acyclic cofibrations.*

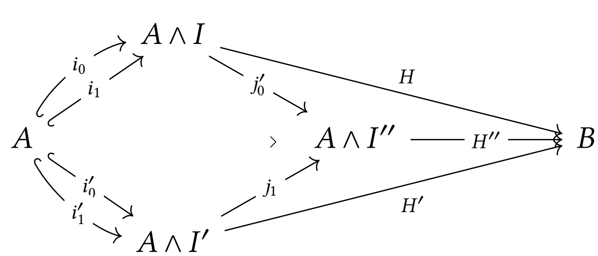
Proof. This is simply because if A is cofibrant, then by the pushout diagram of $\varnothing \to A$, we know $A \to A \coprod A$ is a cofibration, and id_A factors as $A \overset{i_0}{\to} A \wedge I \overset{\sim}{\to} A$, so i_0 is weak equivalence, the same for i_1. □

Proposition 1.11. *If A is cofibrant, then $\overset{l}{\sim}$ defines an equivalence relation on $\mathrm{Hom}_C(A, B)$.*
Dually, If B is fibrant, then $\overset{r}{\sim}$ defines an equivalence relation on $\mathrm{Hom}_C(A, B)$.

Proof. We only prove for the first case that A is cofibrant. It is easy to see that $f \overset{l}{\sim} f$ as we can take A itself as a cylinder object of A and f is the homotopy from f to f. Also if $f \overset{l}{\sim} g$, then switch the maps i_0 and i_1 we get $g \overset{l}{\sim} f$ easily. Now assume that we have $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$. Suppose $f \overset{l}{\sim} g$ via good cylinder object $A \wedge I$ and $g \overset{l}{\sim} h$ via good cylinder object $A \wedge I'$ so that

$$\begin{aligned} g = Hi_1 &: A \overset{i_1}{\hookrightarrow} A \wedge I \overset{H}{\to} B, \\ = Hi'_0 &: A \overset{i'_0}{\hookrightarrow} A \wedge I' \overset{H'}{\to} B \end{aligned}$$

where i_0, i_1, i'_0, i'_1 are acyclic cofibrations by Lemma 1.10 (we use the cofibrant condition here).
Let $A \wedge I''$ be the pushout of $A \wedge I' \hookleftarrow A \hookrightarrow A \wedge I$. We have



where the middle square is the pushout diagram, by Proposition 1.4we know that j_0' and j_1 are acyclic cofibrations, hence $A \to A \wedge I''$ is a cylinder object of A. Moreover, by universal property we have $H'' j_0' = H, H'' j_1 = H'$, this gives the homotopy from f to h we want. $\square$

Notation: If A is cofibrant, we use $\pi^l(A, B)$ to denote the equivalent classes under left homotopy in $\mathrm{Hom}_C(A, B)$. If B is fibrant, we use $\pi^r(A, B)$ to denote the equivalent classes under right homotopy in $\mathrm{Hom}_C(A, B)$.
When A is cofibrant and B is fibrant, by Lemma 1.9 we use $\sim$ to denote two maps are homotopic, and $\pi(A, B)$ the set of equivalent classes under homotopies.

Theorem 1.12. *Let $A, X \in C$ be both cofibrant and fibrant, then for $f \in \mathrm{Hom}_C(A, X)$, f is a weak equivalence if and only if there exists $g : X \to A$ such that $fg \sim \mathrm{id}_X, gf \sim \mathrm{id}_A$. If so, we say g a homotopy inverse of f.*

Proof. Assume first that f is a weak equivalence, using **MC5** to factorize f as: $A \overset{q}{\hookrightarrow} C \overset{p}{\twoheadrightarrow} X$. We can deduce that both q, p are weak equivalences. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ C & \longrightarrow & * \end{array}$$

we have a lifting $r : C \to A$ with $rq = \mathrm{id}_A$, as q is acyclic cofibration and A is fibrant. We claim that $qr \sim \mathrm{id}_C$.

Lemma 1.13. *If A is cofibrant and $p : X \to Y$ is an acyclic fibration, then the pushforward: $p_* : \pi^l(A, X) \to \pi^l(A, Y), [f] \mapsto [pf]$ induces a bijection.*
If X is fibrant and $q : A \to B$ is an acyclic cofibration, then the pullback: $q_* : \pi^l(B, X) \to \pi^l(A, X), [f] \mapsto [fq]$ induces a bijection.

Now C is fibrant since $C \twoheadrightarrow X$ and X is fibrant (also C is cofibrant). By Lemma above, the pullback q^* induces $\pi^r(C, C) \leftrightarrow \pi^r(A, C)$ so that $q^*([qr]) = [qrq] = [q] = q^*([\mathrm{id}_C])$, hence $[qr] \sim \mathrm{id}_C$ as we want. The same machinery applies to prove that there is $s : X \to C$ such that $ps \sim \mathrm{id}_C, sp \sim \mathrm{id}_X$. And the composition $rs : X \to A$ is a homotopy inverse of f.
Conversely, assume f has a homotopy inverse g. And we assume in the decomposition $A \overset{q}{\hookrightarrow} C \overset{p}{\twoheadrightarrow} X$, q is an acyclic cofibration. It remains to show that p is a weak equivalence. Let $H : X \wedge I \to X$ be a good homotopy between fg and id_X, consider the diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{qg} & C \\ & & {\scriptstyle i_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{i_1} & X \wedge I & \xrightarrow[H]{} & X \end{array}$$

By Lemma 1.10, i_0, i_1 are acyclic cofibrations, so there is a lifting $H' : X \wedge I \to C$. Set $s := H'i_1$, then $ps = pH'i_1 = \mathrm{id}_X$. Throughout the proof above, we know $q : A \xrightarrow{\sim} C$ has a homotopy inverse r. Since $pqr = fr$, and $qr \sim \mathrm{id}_C$, we have $p \sim fr$. And $s \sim qg$ via the homotopy H'. We get

$$sp \sim qgp \sim qgfr \sim qr \sim \mathrm{id}_C$$

So sp is a weak equivalence. Also p is a retract of sp, by axiom **MC3**, we get that p is a weak equivalence. □

Definition 1.14. We define the following categories:
$\pi\mathcal{C}_\mathrm{c}$: A category has same objects as $\mathcal{C}_\mathrm{c}$, maps are right homotopy classes of maps in $\mathcal{C}_\mathrm{c}$.
$\pi\mathcal{C}_\mathrm{f}$: A category has same objects as $\mathcal{C}_\mathrm{f}$, maps are left homotopy classes of maps in $\mathcal{C}_\mathrm{f}$.
$\pi\mathcal{C}_\mathrm{cf}$: A category has same objects as $\mathcal{C}_\mathrm{cf}$, maps are homotopy classes of maps in $\mathcal{C}_\mathrm{cf}$.
And functors: $R : \mathcal{C} \longrightarrow \pi\mathcal{C}_\mathrm{f}, X \mapsto RX; \quad Q : \mathcal{C} \longrightarrow \pi\mathcal{C}_\mathrm{c}, X \mapsto QX$ as in Remark 1.3.

Lemma 1.15. *The categories and functors in Definition 1.14 are well defined. Concretely we have:*
(i) If A is cofibrant, then the composition of maps in $\mathcal{C}$ induces $\pi^r(A,B) \times \pi^r(B,C) \to \pi^r(A,C)$. If X is fibrant, then the composition of maps in $\mathcal{C}$ induces $\pi^l(Z,Y) \times \pi^l(Y,X) \to \pi^l(Z,X)$.
(ii) R, Q are functors. Moreover, the restriction of R to $\mathcal{C}_\mathrm{c}$ induces $R' : \pi\mathcal{C}_\mathrm{c} \to \pi\mathcal{C}_\mathrm{cf}$.

Definition 1.16. We define the **Homotopy category** of a model category $\mathcal{C}$ to be the category $\mathrm{Ho}(\mathcal{C})$ so that it has the same objects as in $\mathcal{C}$ and the morphisms are given by

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) := \mathrm{Hom}_{\pi\mathcal{C}_\mathrm{cf}}(R'QX, R'QY) = \pi(RQX, RQY)$$

And let $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the localization functor that keeps the objects in $\mathcal{C}$ and maps a morphism $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$ to $R'Q(f) \in \mathrm{Mor}(\mathrm{Ho}(\mathcal{C}))$.

We mentioned in the definition of γ that it could be regarded as a localization functor, in fact, the construction of homotopy category $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ is canonical, see the following Proposition.

Proposition 1.17. *Let W be the class of weak equivalences, then there is an equivalence of categories $\mathcal{C}[\mathsf{W}^{-1}] \cong \mathrm{Ho}(\mathcal{C})$. In particular, a morphism f in $\mathcal{C}$ is in W if and only if $\gamma(f)$ is an isomorphism.*

1.3 Examples

The following method in [DS95] is crucial to construct model category structures on our favorite categories.
We assume that $\mathcal{C}$ is a cocomplete category. Let $\mathbb{F} = \{f_i : A_i \to B_i\}_{i \in \mathcal{I}}$ be a family of maps in $\mathcal{C}$ and let

$p : X \to Y$ be a map in $\mathcal{C}$. Denote by $S(i)$, index by $i \in \mathcal{I}$, the set of pairs (g, h), $g : A_i \to X, h : B_i \to Y$ so that the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{g} & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p} \\ B_i & \xrightarrow[h]{} & Y \end{array}$$

We define $G^1(\mathbb{F}, p) \in \mathcal{C}$ to be the pushout:

$$\begin{array}{ccccc} \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} A_i & \xrightarrow{\sum_{i\in\mathcal{I}} i \sum_{(g,h)\in S(i)} g} & X & & \\ {\scriptstyle \coprod_{i\in\mathcal{I}} f_i}\downarrow & & \downarrow{\scriptstyle i_1} & \searrow{\scriptstyle p := p_0} & \\ \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} B_i & \longrightarrow & G^1(\mathbb{F}, p) & & \\ & \searrow{\scriptstyle \sum_{i\in\mathcal{I}} i \sum_{(g,h)\in S(i)} h} & & \searrow{\scriptstyle p_1} & \\ & & & & Y \end{array}$$

where we denote i_1 the map $X \to G^1(\mathbb{F}, p)$ and $p_1 : G^1(\mathbb{F}, p) \to Y$ is determined by universal property for the map $p \circ (\sum_{i\in\mathcal{I}} i \sum_{(g,h)\in S(i)} g) = \sum_{i\in\mathcal{I}} i \sum_{(g,h)\in S(i)} h$. We now replace X by $G^1(\mathbb{F}, p)$ and $p := p_0$ by p_1, repeat the above process, we get $G^1(\mathbb{F}, p_1)$, denote by $G^2(\mathbb{F}, p)$. We let $G^k(\mathbb{F}, p) := G^1(\mathbb{F}, p_{k-1})$ for the sequel, then:

$$\begin{array}{ccccccccccc} X & \xrightarrow{i_1} & G^1(\mathbb{F}, p) & \xrightarrow{i_2} & G^2(\mathbb{F}, p) & \longrightarrow & \dots & \longrightarrow & G^k(\mathbb{F}, p) & \longrightarrow & \dots \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & & & {\scriptstyle p_k}\downarrow & & \\ Y & = & Y & = & Y & = & \dots & = & Y & = & \dots \end{array}$$

We take $G^\infty(\mathbb{F}, p) := \operatorname{colim}_k G^k(\mathbb{F}, p)$ and $i_\infty : X \to G^\infty(\mathbb{F}, p)$, $p_\infty : G^\infty(\mathbb{F}, p) \to Y$ so that $p_\infty i_\infty = p$.

Definition 1.18. We say an object $A \in \mathcal{C}$ is **sequentially small** if for any functor $\mathbf{B} : \mathbb{Z}_+ \to \mathcal{C}$, the canonical map $\operatorname{colim}_n \operatorname{Hom}(A, \mathbf{B}(n)) \longrightarrow \operatorname{Hom}(A, \operatorname{colim}_n \mathbf{B}(n))$ is an isomorphism.

Proposition 1.19. *Suppose that for all $i \in \mathcal{I}$, $A_i \in \mathcal{C}$ is sequentially small, then $p_\infty : G^\infty(\mathbb{F}, p) \to Y$ has right lifting property with respect to any $f_i \in \mathbb{F}$.*

Proof. We need a lifting $B_i \to G^\infty(\mathbb{F}, p)$ in the left of following diagrams:

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^\infty(\mathbb{F},p) \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y \end{array} \qquad \begin{array}{ccccccc} A_i & \xrightarrow{g'} & G^k(\mathbb{F},p) & \xrightarrow{i_{k+1}} & G^{k+1}(\mathbb{F},p) & \longrightarrow & G^\infty(\mathbb{F},p) \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_{k+1}} & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y & = & Y & = & Y \end{array}$$

Since A_i is sequentially small, there exist $k \in \mathbb{Z}_+$ and $g' : A_i \to G^k(\mathbb{F}, p)$ such that g is the composition of $G^k(\mathbb{F}, p) \to G^\infty(\mathbb{F}, p)$ with g'. Now $(g', h) \in S^{(k)}(i)$ is a pair for the map $p_k : G^k(\mathbb{F}, p) \to Y$, so there exists a map $B_i \to G^{k+1}(\mathbb{F}, p)$, as depicted in the right of above diagrams. The squares in this diagram are all commutative, thus composing with the map of $G^{k+1}(\mathbb{F}, p) \to G^\infty(\mathbb{F}, p)$, we obtain the lifting we want. □

Example 1.20. We are now ready to construct the model category structure for $\mathbf{Ch}^{\geqslant 0}(R)$ the category of chain complex of R-modules in nonnegative degrees, where R any associative ring. We note that for $M_\bullet \in \mathbf{Ch}^{\geqslant 0}(R)$, it is sequentially small if and only if it has finitely many degrees k so that M_k are nonzero and those nonzero R-module are finitely presented.
To exhibit $\mathbf{Ch}^{\geqslant 0}(R)$ as a model category, we set the class $(W), (C), (F)$ in $\mathbf{Ch}^{\geqslant 0}(R)$ to be:
(W): the weak equivalences are those maps $f : M_\bullet \to N_\bullet$ such that f induces isomorphisms on each degree $f_k : H_k(M_\bullet) \cong H_k(N_\bullet), \forall k \geqslant 0$.
(C): the cofibrations are those maps $f : M_\bullet \to N_\bullet$ such that $f_k : M_k \hookrightarrow N_k$ monomorphism with cokernel being projective R-modules on each degree $k \geqslant 0$.
(F): the fibrations are those maps $f : M_\bullet \to N_\bullet$ such that $f_k : M_k \twoheadrightarrow N_k$ epimorphisms on each degree $k > 0$.
We only prove that **MC5** is satisfied. For $n \geqslant 1$, let $D_n(-) : R\,\mathrm{Mod} \to \mathbf{Ch}(R)$ be the functor:

$$A \in R\,\mathrm{Mod} \mapsto [\cdots \longrightarrow 0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow \dots], \text{ concentrates in degrees } n-1 \text{ and } n.$$

Also for $n \geqslant 0$, define the functor $K(-, n) : A \in R\,\mathrm{Mod} \mapsto M_\bullet$ complex concentrates at degree n with $M_n = A$.

Lemma 1.21. *(i) $D_n(-)$ is left adjoint to the n-th truncation functor, and $K(-, n)$ is left adjoint to the n-th homology functor. Namely we have:*

$$\mathrm{Hom}_{\mathrm{Ch}^{\geqslant 0}(R)}(D_n(A), M_\bullet) = \mathrm{Hom}_{R\,\mathrm{Mod}}(A, M_n); \quad \mathrm{Hom}_{\mathrm{Ch}^{\geqslant 0}(R)}(K(A, n), M_\bullet) = \mathrm{Hom}_{R\,\mathrm{Mod}}(A, H_n(M_\bullet)).$$

(ii) Let $D_n := D_n(R)$ and $S^n := K(R, n)$, then a map $X_\bullet \to Y_\bullet$ is a fibration in the above sense if and only if it has RLP with respect to $0 \to D_n$, for all $n \geqslant 1$. In addition, it is an acyclic fibration if and only if it has RLP with respect to $S^{n-1} \to D_n$, for all $n \geqslant 1$.

Now for any map $f : X_\bullet \to Y_\bullet$, let the family be $\mathbb{F} := \left\{ j_n : S^{n-1} \to D_n \right\}_{n \geqslant 1}$. By construction, we can factor f as $X_\bullet \xrightarrow{i_\infty} G^\infty(\mathbb{F}, f) \xrightarrow{p_\infty} Y_\bullet$. We know that S^i is sequentially small, $\forall i \geqslant 0$, then by Proposition 1.19 and Lemma 1.21, we know that p_∞ is an acyclic fibration.
Consider the following diagram in $\mathbf{Ch}^{\geqslant 0}(R)$:

$$\begin{array}{ccc} \coprod_{n\geqslant 1} S^{n-1} & \longrightarrow & G^k(\mathbb{F}, f) \\ {\scriptstyle j_n}\downarrow & & \downarrow \\ \coprod_{n\geqslant 1} D_n & \longrightarrow & G^{k+1}(\mathbb{F}, f) \end{array}$$

We can see that on each degree n, $G^{k+1}(\mathbb{F}, f)_n = G^k(\mathbb{F}, f)_n \oplus (\bigoplus_{\text{several}} R)$. Passing to infinity, we know $G^\infty(\mathbb{F}, f)$ is the direct sum of X_n with many copies of R, the shows $X_\bullet \xrightarrow{i_\infty} G^\infty(\mathbb{F}, f)$ is a cofibration and thus prove **MC5**(i).
Similarly, take another family $\mathbb{F}' := \left\{ j_n' : 0 \to D_n \right\}_{n\geqslant 1}$, we can factor f as $X_\bullet \xrightarrow{i_\infty'} G^\infty(\mathbb{F}', f) \xrightarrow{p_\infty'} Y_\bullet$. Now p_∞' is a fibration and i_∞' is an acyclic cofibration. Hence **MC5**(ii) is also proved.

Example 1.22. The category **Top** of topological spaces also carries a model category structure, we set
(W): a map $f : X \to Y$ of topological spaces is in W if it is a weak homotopy equivalence.
(F): a map $f : X \to Y$ of topological spaces is a fibration if it is a *Serre fibration*, i.e. it has RLP with respect to $A \times \{0\} \hookrightarrow A \times [0, 1]$ for any CW complex A.
(C) a map $f : X \to Y$ of topological spaces is a cofibration if it has LLP with respect to all acyclic fibrations.
Then the axioms **MC** are satisfied due to several facts.

Lemma 1.23. *(i) Assume that $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ is a sequence of closed subspaces, so that for any $i \geqslant 0$, (X_{i+1}, X_i) is a relative CW pair, then for any finite CW complex A, we have:*

$$\operatorname{colim}_n \operatorname{Hom}_{\mathbf{Top}}(A, X_n) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{Top}}(A, \operatorname{colim}_n X_n)$$

(ii) In the above situation, we say the map $X_0 \to \operatorname{colim}_n X_n$ is a generalized relative CW inclusion, then any cofibration in **Top** *is a retract of a generalized relative CW inclusion.*
(iii) Similar to Lemma 1.21(ii), a map $f : X \to Y$ is a Serre fibration if it has RLP with respect to $D_n \times \{0\} \hookrightarrow D_n \times [0, 1], \forall n \geqslant 1$, here D_n are genuine disk of dimension n. It is moreover a weak equivalence if it has RLP with respect to $S^{n-1} \to D_n, \forall n \geqslant 1$.

Take $\mathbb{F} := \{j_n : D_n \times \{0\} \to D_n \times [0, 1]\}_{n\geqslant 1}$ a family of maps. As before, any map $f : X \to Y$ factors as

$X \xrightarrow{i_\infty} G^\infty(\mathbb{F}, f) \xrightarrow{p_\infty} Y$. For any $p : M \to N$ Serre fibration, we have

$$\begin{array}{ccccc} \coprod_n \coprod_{S(n)} D_n & \longrightarrow & X & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle i_1} & \nearrow & \downarrow{\scriptstyle p} \\ \coprod_n \coprod_{S(n)} D_n \times [0,1] & \longrightarrow & G^1(\mathbb{F}, f) & \longrightarrow & N \end{array}$$

where the lifting $\coprod_n \coprod_{S(n)} D_n \times [0,1] \to M$ exists since $p : M \to N$ is a Serre fibration. And by universal property of pushout, i_1 has LLP with respect to $p : M \to N$. And by construction, $G^1(\mathbb{F}, f)$ is homotopy equivalent to X, hence i_1 a weak equivalence. Passing to the colimit, we know that i_∞ is in (W) ∩ (C). And p_∞ is a Serre fibration due to Proposition 1.19 and Lemma 1.23. We thus construct the factorization for **MC5** (ii). **MC5** (i) is similarly proved.

As a corollary of **Top** being a model category, we can compute the homotopy class $\pi(A, X)$ from a CW complex A to an arbitrary topological space X, is the set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(A, X)$, simply because $A \times I$ is a good cylinder object of A.

1.4 Quillen's adjunction

Let $\mathcal{C}$ be a model category and $F : \mathcal{C} \to \mathcal{D}$ a functor. We first introduce the notions of *left/right derived functors* of F.

Definition 1.24. As above, $F : \mathcal{C} \to \mathcal{D}$ functor with $\mathcal{C}$ model category. The **left derived functor** of F is a pair (LF, t) universal by left among pairs (G, s), where $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$, $s : G \circ \gamma \to F$ a natural transform. The universality of (LF, t) (if exists) says there is a unique natural transform $s' : G \to LF$ such that the composition $G \circ \gamma \xrightarrow{s' \circ \gamma} (LF) \circ \gamma \xrightarrow{t} F$ is $s \circ \gamma$.

Similarly, the **right derived functor** of F is a functor $RF : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ universal by right.

Proposition 1.25. *Let $\mathcal{C}$ be a model category, suppose $F : \mathcal{C} \to \mathcal{D}$ sends all weak equivalences between cofibrant objects to isomorphisms, then the left derived functor LF exists.*

Proof. We first prove that the restriction of F to $\mathcal{C}_c$ identifies right homotopic maps. Assume that $f, g : A \to B$ maps in $\mathcal{C}_c$ such that $f \overset{r}{\sim} g$. Then there exists B^I a very good path object for B and a right homotopy map $H : A \to B^I$. Let $p_i : B^I \to B$ be the composition of $B^I \to B \times B \xrightarrow{p_i} B, i = 0, 1$, we know $f = p_0 H, g = p_1 H$ and let $\omega : B \to B^I$ be the structure map of path objects, we also have $p_i \circ \omega = \mathrm{id}_B$. The map ω is acyclic, by assumption B is cofibrant, so B^I is also cofibrant, hence $F(\omega)$ is an isomorphism. This shows $F(p_0) = F(p_1)$ and thus $F(f) = F(g)$.

So the restriction of F to $\mathcal{C}_c$ induces a functor $F' : \pi\mathcal{C}_c \to \mathcal{D}$, where $\pi\mathcal{C}_c$ has same objects as in $\mathcal{C}_c$ and morphisms are right homotopy classes. As before, denote the restriction functor $\mathcal{C} \to \pi\mathcal{C}_c$ by Q, it satisfies the proeperty that for any morphism f in $\mathcal{C}$, if f is a weak equivalence, then $Q(f)$ is

a right homotopy class represented by a weak equivalence in $\mathcal{C}$. So $F'Q$ sends weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. By universal property of localizing, there exists a unique functor $LF : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $LF \circ \gamma = F'Q$. And we let the natural transform t assigns each object $X \in \mathcal{C}$ to $t_X := F(p_X) : F(QX) \to FX$.
For any pair (G, s), the natural transform $s' : G \to LF$ is given by assigning $X \in \mathcal{C}$ to

$$s'_X := s_{QX} \circ G(\gamma(p_X))^{-1} : GX \to G(QX) \to F(QX) = LF(X)$$

It is not hard to verify that s' is well-defined and that $t \circ (s'\gamma)$ recovers s. □

Definition 1.26. Suppose both $\mathcal{C}, \mathcal{D}$ are model categories. Let $F : \mathcal{C} \to \mathcal{D}$ a functor, then a **total left derived functor** of F is a functor $\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$, defined as the left derived functor of $\gamma_D \circ F : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$.
Similarly, a **total right derived functor** of F is a functor $\mathbb{R}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$, defined as the right derived functor of $\gamma_D \circ F : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$.

Theorem 1.27 (Quillen)**.** *Let $\mathcal{C}, \mathcal{D}$ be two model categories and assume that $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ adjoint pair of functors, then:*
(i) Assume that F preserves cofibrations and G preserves fibrations, then $\mathbb{L}F$ and $\mathbb{R}G$ exists and form an adjoint pair:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \Longleftrightarrow \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$$

(ii) Based on the assumption of (i), if we in addition suppose that for any cofibrant object $A \in \mathcal{C}$ and fibrant object $X \in \mathcal{D}$, $f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)$ is a weak equivalence if and only if its adjoint $f' \in \mathrm{Hom}_{\mathcal{C}}(A, G(X))$ is, then: $\mathbb{L}F$ and $\mathbb{R}G$ are inverse equivalences between homotopy categories.

Proof of Theorem 1.27. To use Proposition 1.25, we would like to first prove that the composition $\gamma_D \circ F$ sends weak equivalence between cofibrant objects to isomorphisms in $\mathrm{Ho}(\mathcal{D})$.
Note that the condition in (i) is equivalent to say that F preserves cofibrations and acyclic cofibrations or (by adjunction) that G preserves fibrations and acyclic fibrations. In fact, assume $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an acyclic cofibration and $g : X \to Y$ any fibration map. If F preserves acyclic cofibration, then $X \to Y$ has RLP with respect to $F(A) \overset{\cong}{\hookrightarrow} F(B)$ (as in the right diagram)

$$\begin{array}{ccc} A & \xrightarrow{u} & G(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(g)} \\ B & \xrightarrow[v]{} & G(Y) \end{array} \qquad\qquad \begin{array}{ccc} F(A) & \xrightarrow{u'} & X \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle g} \\ F(B) & \xrightarrow[v']{} & Y \end{array}$$

By adjunction, the commutative diagram in the left admits a lifting, this shows the equivalence claimed.

Similar to the first step in the proof of Proposition 1.25, we can prove

Lemma 1.28. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor between model categories that sends acyclic cofibrations between cofibrant objects into weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

This shows that $\mathbb{L}F$ exists. The existence of $\mathbb{R}G$ is showed by dual argument for fibrations and fibrant objects.

Now assume $A \in \mathcal{C}$ is cofibrant and $X \in \mathcal{D}$ is fibrant, then $G(X)$ is fibrant since G preserves fibrations. If $f, g : A \to G(X)$ are in the same homotopy class, there exists a good cylinder object $A \wedge I$ and a homotopy $H : A \wedge I \to G(X)$. One can verify that $F(A \wedge I) \to F(A)$ is again a good cylinder object of $F(A)$, and by adjunction this gives a homotopy between f' and $g' : F(A) \to X$. Combining its dual argument, this shows there is a bijection $\pi(A, GX) \cong \pi(FA, X)$ when A is cofibrant and X is fibrant.

To show that $(\mathbb{L}F, \mathbb{R}G)$ is an adjunction pair, we want a bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, \mathbb{R}G(X)) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(\mathbb{L}F(A), X)$$

We only need to show this for A cofibrant and X fibrant. Recall the constructions

$$\emptyset \hookrightarrow QA \overset{p_A}{\twoheadrightarrow} A, \quad X \overset{i_X}{\hookrightarrow} RX \twoheadrightarrow *$$

we have

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, \mathbb{R}G(X)) \overset{(\gamma(p_A))^*}{\longrightarrow} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QA, G(R(X))) = \pi(A, GX) \quad G(R(X)) = R(G(X)) \\ \cong &\pi(F(A), X) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(F(Q(A)), RX) \overset{(\gamma(i_X)^{-1})_*}{\longrightarrow} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(\mathbb{L}F(A), X) \quad F(Q(A)) = Q(F(A)). \end{aligned}$$

This proves (i) of the Theorem. Now assume the condition in (ii) holds, then $\epsilon_A : A \to \mathbb{R}G(\mathbb{L}F(A))$ map in $\mathrm{Ho}(\mathcal{C})$ which is adjoint to $\mathrm{Id}_{\mathbb{L}F(A)}$. We know ϵ_A is an isomorphism by above formula, and ϵ gives a natural transform from Id to $\mathbb{R}G\mathbb{L}F$. And there is a version for $\mathbb{L}F\mathbb{R}G$, we showed the equivalence of (ii). □

Example 1.29 (An ∞-categorical example)**.** Let $\mathbf{Set}_\Delta$ be the category of simplicial sets, i.e. category of functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathbf{\Delta}$ has objects being ordered sets $[n] = \{0 < 1 ... < n\}$, and morphisms are order-preserving maps. For $n \geqslant 1$ and $0 \leqslant i \leqslant n$, we have a canonical order preserving bijection $[n-1] \cong [n] \backslash \{i\}$ and an inclusion $d_i^n : [n-1] \hookrightarrow [n]$. Let $S_\bullet \in \mathbf{Set}_\Delta$, the map d_i^n induces a map $S_n \to S_{n-1}$ called the **face map**. Similarly for $n \geqslant 0$ and $0 \leqslant i \leqslant n$, we have a canonical surjection $s_i^n : [n+1] \longrightarrow\!\!\!\!\!\rightarrow [n]$ which is constant on $\{i, i+1\}$, the induced map $s_i^n : S_n \to S_{n+1}$ is called the **degeneracy map**.

Let X be a topological space, we can associate it with a simplicial set $\mathrm{Sing}_\bullet(X)$:

For each $[n] \in \mathbf{\Delta}$, we assign $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \in \mathbf{Set}$, where Δ^n is the n-simplex. And for each $\alpha : [m] \to [n]$ non-decreasing map, we assign it with $|\Delta^m| \xrightarrow{\alpha_*} |\Delta^n|$ given by

$$|\Delta^m| \to |\Delta^n| : (t_0, ..., t_1) \mapsto \left(\sum_{\alpha(i)=0} t_i, \sum_{\alpha(i)=1} t_i, ..., \sum_{\alpha(i)=n} t_i \right)$$

which induces a morphism $\mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X)$. Moreover, for $X_\bullet \in \mathbf{Set}_\Delta$, there is a geometric realization functor: $|\bullet| : s\mathbf{Set} \to \mathbf{Top}, X_\bullet \mapsto |X_\bullet|$, see for example [Lur, Tag 001X]. And there is an adjunction pair:

$$|\bullet| : \mathbf{Set}_\Delta \Longleftrightarrow \mathbf{Top} : \mathrm{Sing}_\bullet$$

We can establish a model category structure on $\mathbf{Set}_\Delta$ by letting $f : X_\bullet \to Y_\bullet$ map of simplicial sets be
(W): weak equivalence if $|f|$ is a weak homotopy equivalence in **Top**.
(C): cofibration if $X[n] \to Y[n], n \geqslant 0$ is a monomorphism.
(F): fibration if f has RLP with respect to acyclic cofibrations.
Then Quillen shows the assumptions (i) and (ii) are satisfied, with respect to the model category structures on $\mathbf{Set}_\Delta$ and **Top** (as in previous section). Hence there is an equivalence between Ho(**Top**) and Ho($\mathbf{Set}_\Delta$). More importantly, the notion of ∞-**category** comes from a certain class in $\mathbf{Set}_\Delta$.

For any $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$, we let Δ^n and $\Lambda^n_i \in \mathbf{Set}_\Delta$ to be

$$\Delta^n : \Delta^{op} \to \mathbf{Set}[m] \mapsto \mathrm{Hom}_\Delta([m],[n]) \quad \text{note that } |\Delta^n| \text{ is the geometric n-simplex;}$$
$$\Lambda^n_i : [m] \mapsto \left\{ \alpha \in \mathrm{Hom}_\Delta([m],[n]) \mid [n] \nsubseteq \alpha([m]) \cup \{i\} \right\} \qquad i\text{-th horn in } \Delta^n$$

Definition 1.30. We say $S_\bullet \in \mathbf{Set}_\Delta$ is a **Kan complex** if it satisfies the following Kan condition:

[Kan] Any simplicial morphism $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be lift to a simplicial morphism $\Delta^n \to S_\bullet$ for any $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$.

We say $S_\bullet \in \mathbf{Set}_\Delta$ is an $(\infty, 1)$-**category** or a **weak Kan complex** if it satisfies the following weak Kan condition:

[weak Kan] Any simplicial morphism $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be lift to a simplicial morphism $\Delta^n \to S_\bullet$ for any $n \in \mathbb{N}$ and $0 < i < n$.

Proposition 1.31. *Let $X \in \mathbf{Top}$ be a topological space, then $\mathrm{Sing}_\bullet(X)$ is an $(\infty, 1)$-category.*

Definition 1.32. For an ordinary small category C, define its nerve $\mathsf{N}_\bullet(C) \in \mathbf{Set}_\Delta$ so that $\mathsf{N}_k(C)$ is the set of all functors $[k] \to C$.

Proposition 1.33. *For any small category C, its nerve $\mathsf{N}_\bullet(C)$ is an $(\infty, 1)$-category.*

If there is no ambiguity, we will use the terminology ∞-category instead of $(\infty, 1)$-categorty. Also we write $\mathbf{Cat}_\infty$ the category of ∞-categories, with morphisms to be maps between simplicial sets.
As the name "category" is endowed to a weak Kan complex, we need first figure out its objects and its morphisms. Let C be a weak Kan complex, which is in the form of $S_\bullet$, then we define the set of objects of C to be the set S_0. In the same spirit, the morphisms of C are defined to be S_1.
For any $f \in S_1$, recall we have face map d_0^1 and d_1^1, we put $X := d_1^1(f)$ as the source of f and $Y := d_0^1(f)$ as the target of f. For any $X \in S_0$, there is degeneracy map $s_0^0(X)$, which we called the identity endomorphism of X, denoted by $\mathbf{1}_X$.
Similar to the case of model categories, we can define the homotopy category of ∞-category, which has more topological meaning. Again, let $C = S_\bullet$ be an ∞-category and $X, Y \in S_0$, let $f, g : X \to Y \in S_1$ two maps in C, one can imagine that the homotopies between f and g lie in "higher" simplices. In fact, $\sigma \in S_2$ is called a **homotopy** from f to g (denoted by $f \overset{\sigma}{\sim} g$) if it satisfies

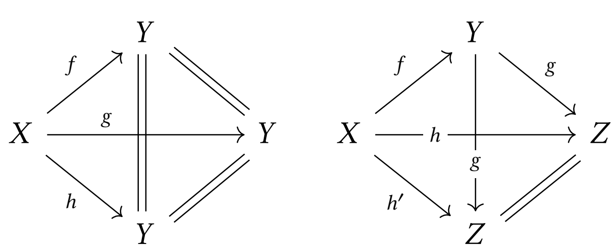
$$d_0^2(\sigma) = \mathbf{1}_Y, d_1^2(\sigma) = g, d_2^2(\sigma) = f.$$

We say f and g with same source and target are **homotopic** if there exists homotopy from f to g. Also, for three objects $X, Y, Z \in C = S_\bullet$ and $f : X \to Y, g : Y \to Z$, we say that a morphism $h : X \to Z$ is a composition of f and g if there exists a $\sigma' \in S_2$ such that

$$d_0^2(\sigma) = g, d_1^2(\sigma) = h, d_2^2(\sigma) = f.$$

Proposition 1.34. *(i) Let $X, Y, Z \in C$ be objects in an ∞-category, denote by $E_{X,Y}$ the subset of 1-simplices of C contains morphisms X to Y. Then homotopic relation defines an equivalence relation on $E_{X,Y}$.*
(ii) For two morphisms $f : X \to Y, g : Y \to Z$, there exists a composition h of f and g and all such compositions are homotopic in $E_{X,Z}$.

Proof. The idea of proof is that if $f \sim g$ and $f \sim h$, want to prove that $g \sim h$. Under this assumption, we have the left following (3,1)-horn of simplex $\Lambda_1^3 \to C$:



By weak Kan condition, we find the bottom 2-simplex exists and gives homotopy $g \sim h$. Similarly for (ii), we can consider the right above simplex. ☐

Thanks to Proposition 1.34, we can define:

Definition 1.35. Let $\mathcal{C}$ be an ∞-category, define its **homotopy category** $h\mathcal{C}$ to be an ordinary category with:

- Objects $\mathrm{Ob}(h\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
- Morphisms $\mathrm{Hom}_{h\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)\Big/ \sim_{\text{homotopy}}$.

We have transitions between (small) categories and ∞-categories. Let $\mathcal{C}$ be an ordinary category, then we can consider $h\mathsf{N}_\bullet(\mathcal{C})$ the homotopy category of Nerve of $\mathcal{C}$. It is not difficult to verify that $h\mathsf{N}_\bullet(\mathcal{C}) \cong \mathcal{C}$. Conversely, starting with an ∞-category $\mathcal{C}$, we can consider $\mathsf{N}_\bullet(h\mathcal{C})$. First, let us construct a morphism of simplicial sets $\mathcal{C} \to \mathsf{N}_\bullet(h\mathcal{C})$. Let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex of $\mathcal{C}$, its vertices give a collection of objects $X_0, X_1, ..., X_n \in \mathrm{Ob}(\mathcal{C})$, its edges give collections of morphisms $f_{ij} : X_i \to X_j, \forall 0 \leqslant i \leqslant j \leqslant n$ and homotopy classes $[f_{ij}] \in \mathrm{Hom}_{h\mathcal{C}}(X_i, X_j)$. Those data induce a functor $[n] \to h\mathcal{C}$ and thus an n-simplex in $\mathsf{N}_\bullet(h\mathcal{C})$, say $u(\sigma)$. We obtain thus a morphism of simplicial sets

$$u : \mathcal{C} \to \mathsf{N}_\bullet(h\mathcal{C}); \quad \sigma \mapsto u(\sigma)$$

Proposition 1.36. *For any test category $\mathcal{D}$, we have the following composition of maps gives a bijection:*

$$\mathrm{Hom}_{\mathbf{Cat}}(h\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbf{Set}_\Delta}\left(\mathsf{N}_\bullet(h\mathcal{C}), \mathsf{N}_\bullet(\mathcal{D})\right) \xrightarrow{u\circ -} \mathrm{Hom}_{\mathbf{Set}_\Delta}\left(\mathcal{C}, \mathsf{N}_\bullet(\mathcal{D})\right)$$

1.5 Simplicial Categories

In Example 1.29, we have encountered the notion of $\mathbf{Set}_\Delta$ and we know that an $(\infty, 1)$-category is a simplicial set with extra properties (weak Kan complex). In this section, we would use the notion of localization to consider the "underlying" $(\infty, 1)$-category structure of Model categories.

Definition 1.37. A **simplicial category** is a category enriched over $\mathbf{Set}_\Delta$. We denote by $\mathbf{Cat}_\Delta$ the category of small simplicial categories, where the morphisms are given by simplicial functors.

We can associate $\mathbf{Cat}_\Delta$ class of weak equivalences as following:
Let $\mathcal{C}_\bullet, \mathcal{D}_\bullet \in \mathbf{Cat}_\Delta$ and $f : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ simplicial functor is said to be a weak equivalence (also known as **Dwyer-Kan equivalence**) if

- For any $x, y \in \mathcal{C}_\bullet$, the induced map $\mathrm{Hom}_{\mathcal{C}_\bullet}(x, y) \to \mathrm{Hom}_{\mathcal{D}_\bullet}(f(x), f(y))$ is a standard weak equivalence in $\mathbf{Set}_\Delta$ (defined in Example 1.29 above).
- f is essentially surjective, namely the induced functor $\mathrm{Ho}(f)$ is essentially surjective.

The class of fibrations in $\mathbf{Cat}_\Delta$ are those $f : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ such that

- For any $x, y \in \mathcal{C}_0$, the induced map $\mathrm{Hom}_{\mathcal{C}_\bullet}(x, y) \to \mathrm{Hom}_{\mathcal{D}_\bullet}(f(x), f(y))$ is a fibration in $\mathbf{Set}_\Delta$.
- The induced functor $\mathrm{Ho}(f)$ is an isofibration, namely any equivalence $f(x) \to y$ in $\mathrm{Ho}(\mathcal{D}_\bullet)$ can be lift to an equivalence $x \to \tilde{y}$ in $\mathrm{Ho}(\mathcal{C}_\bullet)$.

Remark 1.38. Under the above choice of class of fibrations, we can see the isofibration property ensures that fibrant objects in $\mathbf{Cat}_\Delta$ are those categories enriched over Kan complexes, which is also called **locally Kan** (see [Lur, Definition 00JY]).

Theorem 1.39. *[Lur09, Proposition A.3.2.4] There exists a model category structure, called* ***Bergner model structure*** *on* $\mathbf{Cat}_\Delta$ *given by class of weak equivalences and class of fibrations as above.*

We denote by $\mathbf{Cat}_{\mathbf{Kan}}$ the category of locally Kan simplicial categories, with fibrant replacement functor $R : \mathbf{Cat}_\Delta \to \mathbf{Cat}_{\mathbf{Kan}}$ as in Definition 1.14. as in Definition 1.14.
Let $\mathcal{C}_\bullet$ be a simplicial category, we associate it with a simplicial set $\mathsf{N}^{\mathsf{hc}}_\bullet(\mathcal{C})$ called **homotopy coherent nerve** of $\mathcal{C}_\bullet$, given by

$$[n] \in \Delta^{\mathrm{op}} \mapsto \mathrm{Map}_{\mathrm{Cat}_\Delta}\left(\mathsf{Path}[n]_\bullet, \mathcal{C}_\bullet\right)$$

where $\mathsf{Path}[n]_\bullet$ is the simplicial path category of partially ordered set $[n]$ (for general construction of simplicial path category of partially ordered set, see [Lur, Notation 00KN]). We then obtain a functor $\mathsf{N}^{\mathsf{hc}}_\bullet(-) : \mathbf{Cat}_\Delta \to \mathbf{Set}_\Delta$.

Remark 1.40. Let $C = C_0$ be the ordinary category of $C_\bullet$, then we have monomorphism $\mathsf{N}_\bullet(C) \hookrightarrow \mathsf{N}^{\mathsf{hc}}_\bullet(C)$. Take $\underline{C}_\bullet$ to be the constant simplicial category of ordinary category C (i.e. enriched by constant simplicial set $\mathrm{Hom}_C(x, y)$), then we have $\mathsf{N}_\bullet(C) = \mathsf{N}^{\mathsf{hc}}_\bullet(\underline{C})$.

Theorem 1.41 (Cordier-Porter)**.** *Let $C_\bullet$ be a simplicial category that is locally Kan, then its homotopy coherent nerve $\mathsf{N}^{\mathsf{hc}}_\bullet(C)$ is an ∞-category.*

Definition 1.42. We write **Kan** the simplicial category of Kan complexes, with natural simplicially enriched structure. Define the ∞-**category of spaces** to be

$$\mathcal{S} := \mathsf{N}^{\mathsf{hc}}_\bullet(\mathbf{Kan})$$

Proposition 1.43. *The simplicial set $\mathcal{S}$ is an ∞-category.*

Proposition 1.44. *For any $S_\bullet \in \mathbf{Set}_\Delta$, there exists a pair uniquely determined upto isomorphism $(C_\bullet, u)$ of simplicial category $C_\bullet$ and $u : S_\bullet \to \mathsf{N}^{\mathsf{hc}}(C_\bullet)$ such that for any $\mathcal{D}_\bullet \in \mathbf{Cat}_\Delta$, there is a bijection*

$$\left\{ \mathit{SimplicialFunctor} : C_\bullet \to \mathcal{D}_\bullet \right\} \cong \mathrm{Map}_{\mathbf{Set}_\Delta}(S_\bullet, \mathsf{N}^{\mathsf{hc}}_\bullet(\mathcal{D}_\bullet))$$

The functor $\mathsf{N}^{\mathsf{hc}}_\bullet : \mathbf{Cat}_\Delta \to \mathbf{Set}_\Delta$ admits a left adjoint: $\mathsf{Path}[-]_\bullet : \mathcal{S}_\bullet \mapsto \mathsf{Path}[S]_\bullet := C_\bullet$, where $C_\bullet$ is the simplicial category associated to $S_\bullet$ as above.

Theorem 1.45. *For the model category structure constructed on $\mathbf{Cat}_\Delta$ above, there exists another model structure on $\mathbf{Set}_\Delta$ called the **Joyal model structure** so that the adjoint pair*

$$\mathsf{Path}[-]_\bullet : \mathbf{Set}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat}_\Delta : \mathsf{N}^{\mathsf{hc}}_\bullet(-)$$

is a Qullien equivalence.

We give here the explicit construction of Joyal model structure of $\mathbf{Set}_\Delta$:

- we say $f_\bullet F : S_\bullet \to T_\bullet \in \mathrm{Mor}(\mathbf{Set}_{\Delta,\mathrm{Joyal}})$ morphism between simplicial sets is a weak equivalence if $\mathsf{Path}[f]_\bullet$ is a Dwyer-Kan equivalence in $\mathbf{Cat}_\Delta$.
- The cofirations in $\mathbf{Set}_{\Delta,\mathrm{Joyal}}$ is given by monomorphisms.
- The fibrants in $\mathbf{Set}_{\Delta,\mathrm{Joyal}}$ are those weak Kan complexes (i.e. ∞-categories).

Combining Theorem 1.41, 1.45 and Quillen's Theorem 1.27 (ii), we can conclude the following slogan:

Slogan: $R\mathsf{Path}[-]_\bullet$ and $R\mathsf{N}^{\mathsf{hc}}_\bullet(-)$ are inverse to each other (upto equivalence of categories) and establish an equivalence between $\mathbf{Cat}_\infty$ and $\mathbf{Cat}_{\mathbf{Kan}}$.

1.6 Localization of Model/Simplicial Categories

We have seen in previous sections that one can construct the homotopy category of a model category in a way of localization. We are going to see the underlying higher homotopy information, i.e. the ∞-category, of a model category. To quote:

Localizing a model category with respect to a class of maps does not mean making the maps into isomorphisms; instead, it means making the images of those maps in the homotopy category into isomorphisms. Since the image of a map in the homotopy category is an isomorphism if and only if the map is a weak equivalence, localizing a model category with respect to a class of maps means making maps into weak equivalences.

Definition 1.46. Let $(\mathcal{C}, \mathsf{W})$ be a weak equivalence category, for objects $X, Y \in \mathcal{C}$, define for each $n \in \mathbb{N}$ a category $\mathsf{Ham}_n\mathcal{C}(X,Y)$ contains objects to be paths of the form

$$X \xleftarrow{\sim} K_1 \to K_2 \xleftarrow{\sim} K_3 \to \dots \to Y$$

with morphisms to be

$$\begin{array}{ccccccc} & & K_1 & \longrightarrow & K_2 & \xleftarrow{\sim} & \dots & & \\ & \swarrow^{\sim} & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & & \searrow & \\ X & & & & & & & & Y \\ & \nwarrow_{\sim} & & & & & & \nearrow & \\ & & L_1 & \longrightarrow & L_2 & \xleftarrow{\sim} & \dots & & \end{array}$$

And we define $L^H(\mathcal{C}, \mathsf{W})(X,Y) \in \mathbf{Set}_\Delta$ as

$$\coprod_n \mathsf{N}_\bullet\left(\mathsf{Ham}_n\mathcal{C}(X,Y)\right) / \sim$$

the coproduct of nerve of $\mathsf{Ham}_n(X,Y)$ and quotienting the equivalence relation generated by inserting or removing identity morphisms and composing composable morphisms. For $X, Y, Z \in \mathcal{C}$, it is clear that one has composition map $L^H(\mathcal{C}, \mathsf{W})(X,Y) \times L^H(\mathcal{C}, \mathsf{W})(Y,Z) \to L^H(\mathcal{C}, \mathsf{W})(X,Z)$.
We thus obtain a simplicially enriched category $L^H(\mathcal{C}, \mathsf{W}) \in \mathbf{Cat}_\Delta$, called the **hammock localization** of $\mathcal{C}$ with respect to W, given by Dwyer-Kan.

Remark 1.47. Recall that for an ordinary (small) weak equivalence category $(\mathcal{C}, \mathsf{W})$, one can define its localization $\mathcal{C}[\mathsf{W}^{-1}]$ to have objects same as $\mathcal{C}$, with "set" of morphisms:

$$\mathrm{Hom}_{\mathcal{C}[\mathsf{W}^{-1}]}(X,Y) := \{X \to K_1 \leftarrow K_2 \to \dots \to Y \mid \text{ all left arrows are in } \mathsf{W}\}.$$

This construction can be generalized to $\mathcal{W} \subseteq \mathcal{C}$ subcategory. We can verify the following:

Proposition 1.48. *We have equivalence of categories* $\mathrm{Ho}\left(L^H(\mathcal{C},\mathsf{W})\right) \cong \mathcal{C}[\mathsf{W}^{-1}]$.

Here for a simplicial category $\mathcal{D}_\bullet$, we take its homotopy category $\mathrm{Ho}(\mathcal{D})$ to have same objects as in $\mathcal{D}_\bullet$ with $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(x,y) = \pi_0(\mathrm{Map}_{\mathcal{D}}(x,y))$.

Proposition 1.49. *Let $\mathcal{C}$ be a model category, recall that we have full subcategories $\mathcal{C}_{\mathrm{c}}, \mathcal{C}_{\mathrm{f}}, \mathcal{C}_{\mathrm{cf}}$ spanned by cofibrants, fibrants and cofibrants-fibrants respectively. Then we have the natural maps*

$$L^H\mathcal{C}_{\mathrm{f}} \longrightarrow L^H\mathcal{C} \longleftarrow L^H\mathcal{C}_{\mathrm{c}}$$

with respect to weak equivalences that are equivalences of simplicial categories.

The ∞-category underlying the model category $\mathcal{C}$ with class of weak equivalences W is defined to be

$$\mathsf{N}^{\mathrm{mod}}_\bullet(\mathcal{C}) := R\mathsf{N}^{\mathsf{hc}}_\bullet\left(L^H(\mathcal{C},\mathsf{W})\right) \qquad \textbf{nerve of model category } \mathcal{C}$$

Moreover, in the category of marked simplicial sets, we have the equivalence

$$\left(\mathsf{N}_\bullet(\mathcal{C}),\mathsf{W}\right) \cong \left(\mathsf{N}^{\mathrm{mod}}_\bullet(\mathcal{C})\right)^\natural$$

Let $\mathcal{W}_\bullet \xrightarrow{f_\bullet} \mathcal{C}_\bullet \in \mathbf{Cat}^{[1]}_\Delta$ be a map of simplicial categories, we can associate it with a simplicial category $\mathcal{C}\left[\mathcal{W}^{-1}\right]_\bullet$ so that $\mathcal{C}\left[\mathcal{W}^{-1}\right]_n := \mathcal{C}_n\left[\mathcal{W}_n^{-1}\right]$. One can also generalize the hammock construction to this map of simplicial categories $\mathcal{W}_\bullet \xrightarrow{f_\bullet} \mathcal{C}_\bullet \in \mathbf{Cat}^{[1]}_\Delta$, denote by $L^H(\mathcal{C}_\bullet,\mathcal{W}_\bullet)$ On the other hand, since $\mathbf{Cat}_\Delta$ itself is a model category, we can define the localization with respect to $f_\bullet$ that is relavent to model structure. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{W}_\bullet & \xrightarrow{f_\bullet} & \mathcal{C}_\bullet \\ {\scriptstyle p_\bullet}\uparrow & & \uparrow{\scriptstyle q_\bullet} \\ \tilde{\mathcal{W}}_\bullet & \xrightarrow{\tilde{f}_\bullet} & \tilde{\mathcal{C}}_\bullet \end{array} \qquad p_\bullet, q_\bullet \text{ cofibrant replacement; } \tilde{f}_\bullet \text{ cofibration}$$

then we define the **Dwyer-Kan localization** of $\mathcal{W}_\bullet \xrightarrow{f_\bullet} \mathcal{C}_\bullet \in \mathbf{Cat}^{[1]}_\Delta$ to be $\tilde{\mathcal{C}}[\tilde{\mathcal{W}}^{-1}]_\bullet$, which is weakly equivalent to $L^H(\mathcal{C}_\bullet,\mathcal{W}_\bullet)$.
We then want to promote Proposition 1.49 to simplicial version. To do that, we need first introduce the notion of simplicial model category (not just a $\mathbf{Set}_\Delta$-enriched model category!).

Definition 1.50. Let $\mathcal{V}$ be a closed monoidal category (i.e. for any $v \in \mathcal{C}$ the tensor functor $-\otimes v : \mathcal{V} \to \mathcal{V}$ admits a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{V}}(v,-) : \mathcal{V} \to \mathcal{V}$).

We say a $\mathcal{V}$-enriched category $\mathcal{C}$ is **tensored** if there exists $-\otimes- : \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ such that

$$\mathrm{Hom}_{\mathcal{C}}(v\otimes c, c') = \underline{\mathrm{Hom}}_{\mathcal{V}}(v, \mathrm{Hom}_{\mathcal{C}}(c,c')) \quad \forall c, c' \in \mathcal{C}, \forall v \in \mathcal{V}$$

We say a $\mathcal{V}$-enriched category $\mathcal{C}$ is **powered** if there exists $-\pitchfork- : \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ such that

$$\mathrm{Hom}_{\mathcal{C}}(c, \pitchfork(v,c')) = \underline{\mathrm{Hom}}_{\mathcal{V}}(v, \mathrm{Hom}_{\mathcal{C}}(c,c')) \quad \forall c, c' \in \mathcal{C}, \forall v \in \mathcal{V}$$

Remark 1.51. In above definition, note that $\mathrm{Hom}_{\mathcal{C}}$ is $\mathcal{V}$-valued Hom of $\mathcal{C}$. The category $\mathbf{Set}_\Delta$ is a closed monoidal category, with Cartesian monoidal structure: $\left(S_\bullet\otimes T_\bullet\right)_n = S_n\times T_n$ and the internal hom is given by $\underline{\mathrm{Hom}}_{\mathbf{Set}_\Delta}(S_\bullet, T_\bullet) : [n]\mapsto \mathrm{Hom}_{\mathbf{Set}_\Delta}(S_\bullet\times\Delta[n], T_\bullet)$.

Definition 1.52. We say a simplicial category $\mathcal{C}_\bullet \in \mathbf{Cat}_\Delta$ is a simplicial model category if:
(i) Its underlying category $\mathcal{C}_0$ has model structure.
(ii) It is tensored and powered over $\mathbf{Set}_\Delta$.
(iii) For any cofibration $X\to Y$ in $\mathbf{Set}_\Delta$ and cofibration $A\to B$ in $\mathcal{C}$, the induced pushout product morphism $A\otimes Y \coprod^{A\otimes X} B\otimes X \to B\otimes Y$ is a cofibration in $\mathcal{C}_0$.

Remark 1.53. The model structure of $\mathbf{Set}_\Delta$ in above definition is taken to be the one as Example 1.29. In particular, we find that $\mathbf{Set}_\Delta$ itself with this model structure is a simplicial model category.

Theorem 1.54. *Let $\mathcal{C}_\bullet$ be a simplicial model category, then we have the following equivalences of simplicial categories:*

$$\mathcal{C}^{\mathrm{cf}}_\bullet \to L^H(\mathcal{C}^{\mathrm{cf}}_\bullet) \to L^H(\mathcal{C}_\bullet) \leftarrow L^H(\mathcal{C}, \mathsf{W})$$

Namely, the two localizations of $\mathcal{C}_\bullet$ through model structure and simplicial structure respectively, coincide. We write $\mathcal{C}_\bullet[\mathsf{W}^{-1}] := \mathsf{N}^{\mathsf{hc}}_\bullet(\mathcal{C}^{\mathrm{cf}}_\bullet)$. In particular, one can find that the ∞-category of spaces, also called ∞-category of ∞-groupoids is

$$\mathcal{S} = \mathbf{Set}_\Delta[\mathsf{W}^{-1}]$$

2 DG Categories

2.1 Definition of DG Categories

The word "dg" is an abbreviation for **differential graded**. Let $\mathbf{k}$ be a commutative ring, then a **dg $\mathbf{k}$-module** is an $\mathbb{Z}$-graded $\mathbf{k}$-module $V = \bigoplus_{n\in\mathbb{Z}} V^n$ plus a differential map $\partial_V : V \to V$, such that ∂_V is of degree -1, i.e. $\partial_V(V_m) \subseteq V_{m-1}$ and $\partial_V^2 = 0$. Equivalently, $(\{V^n\}_{n\in\mathbb{Z}}, \partial)$ is a chain complex of $\mathbf{k}$-modules. The morphisms between DG $\mathbf{k}$-modules are graded morphisms that preserves the differentials.
We denote by $\mathrm{Ch}(\mathbf{k})$ the category of chain complexes in $\mathbf{k}$-modules=DG $\mathbf{k}$-modules and $\mathrm{Ch}(\mathbb{Z})$ the category of chain complexes in abelian groups. Morphisms in $\mathrm{Ch}(\mathbf{k})$ are chain complex maps.
There is a monoidal structure on $\mathrm{Ch}(\mathbf{k})$. For $(V, \partial_V), (W, \partial_W)$, their tensor DG $\mathbf{k}$-module is $(V \otimes W, \partial_{V\otimes W})$ where $(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes_{\mathbf{k}} W_q$ and $\partial_{V\otimes W}(x \otimes y) = \partial_V(x) \otimes y + (-1)^{\deg x} x \otimes \partial_W(y)$.

Definition 2.1. One particular example of DG $\mathbf{k}$-modules is **dg $\mathbf{k}$-algebra**. That is, a graded $\mathbf{k}$-algebra

$$(A_* = \{A_n\}_{n\in\mathbb{Z}}, \partial)$$

whose multiplication map $A \otimes A \to A$ (of degree 0) compatible with a differential $\partial : A_* \to A_{*-1}$. In other words, the differential on A_* would in addition satisfy the Leibniz rule

$$\partial_{A_*}(f \cdot g) = \partial_{A_*}(f) \cdot g + (-1)^{\deg f} f \cdot \partial_{A_*}(g)$$

Definition 2.2. A **DG category** (= **differential graded category**) over $\mathbf{k}$ is a category $\mathcal{A}$ that enriched over the category of chain complex of $\mathbf{k}$-modules.
More explicitly, a DG category $\mathcal{A}$ consists of the data of:

- A collection of objects $\mathrm{Ob}(\mathcal{A})$;
- For any $X, Y \in \mathrm{Ob}(\mathcal{A})$, the morphism is a DG $\mathbf{k}$-module $\mathcal{A}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)_*$;
- For any $X, Y, Z \in \mathrm{Ob}(\mathcal{A})$, the composition law

$$\circ_{Z,Y,X} : \mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$$

  which is linear and satisfies the Leibniz rule: $\partial(g \circ f) = (\partial g) \circ f + (-1)^{\operatorname{def} g} g \circ (\partial f)$.
- For $\forall X \in \mathcal{A}$, there exists a unit $\mathbf{1}_X \in \mathcal{A}(X, X)$ for the composition.

Remark 2.3. We can deduce from definition that the collection of DG categories with one object X is in bijection with DG algebras. Also, one can recover the underlying ordinary category $\mathcal{A}^\circ$ of DG category $\mathcal{A}$ by taking $\mathrm{Ob}(\mathcal{A}^\circ) = \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{A}^\circ}(X, Y)$ are those 0-cycles in $\mathrm{Hom}_{\mathcal{A}}(X, Y)_*$. One can deduce from $\partial(\mathbf{1}_X \circ \mathbf{1}_X) = \partial(\mathbf{1}_X) + \partial(\mathbf{1}_X)$ that $\mathbf{1}_X \in \mathrm{Hom}(X, X)_0$ is a 0-cycle.

To establish a DG category $\mathcal{A}$ with an $(\infty,1)$-categorical structure, we need to construct something similar to $\mathsf{N}^{\mathrm{hc}}_{\bullet}$.

Example 2.4. Let $\mathcal{A}$ be a DG category, define $\mathsf{N}^{\mathrm{dg}}_{\bullet}(\mathcal{A}) \in \mathbf{Set}_{\Delta}$ with $\mathsf{N}^{\mathrm{dg}}_{n}(\mathcal{A})$ contains pairs $\left(\{X_i\}_{0\leqslant i\leqslant n}, \{f_I\}_{I\subseteq[n]}\right)$ where $X_i \in \mathrm{Ob}(\mathcal{A})$ and for each $I = \{i_0 > i_1 > \ldots > i_k\}$, $f_I \in \mathrm{Hom}(X_{i_k}, X_{i_0})_{k-1}$ such that

$$\partial f_I = \sum_{a=1}^{k-1}(-1)^a \left(f_{\{i_0>\ldots>i_a\}} \circ f_{\{i_a>\ldots>i_k\}} - f_{I\setminus\{i_a\}}\right) \in \mathrm{Hom}(X_{i_k}, X_{i_0})_{k-2}$$

For non-decreasing function $\alpha : [n] \to [m]$, define $\alpha^* : \mathsf{N}^{\mathrm{dg}}_{m}(\mathcal{A}) \to \mathsf{N}^{\mathrm{dg}}_{n}(\mathcal{A}) : (\{X_i\}_{0\leq i\leq m}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0\leq j\leq n}, \{g_J\})$ by taking

$$g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective} \\ \mathrm{id}_{X_i} & \text{if } J = \{j_0 > j_1\} \text{ with } \alpha(j_0) = i = \alpha(j_1) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.5. *For DG category $\mathcal{A}$, the simplicial set $\mathsf{N}^{\mathrm{dg}}_{\bullet}(\mathcal{A})$ is an ∞-category.*

Proof. See [Lur, Theorem 00PW]. □

Example 2.6. The ordinary category $\mathrm{Ch}(\mathbf{k})$ of DG $\mathbf{k}$-modules also has a DG enhancement. Let (C_*, δ_C) and $(C'_*, \delta_{C'})$ be two chain complexes of $\mathbf{k}$-modules, we define

$$\mathbf{Ch}_{\mathrm{dg}}(\mathbf{k})(C_*, C'_*) = \bigoplus_{n\in\mathbb{Z}} \mathbf{Ch}_{\mathrm{dg}}(R)\left(C_*, C'_*\right)_n$$

so that $\mathbf{Ch}_{\mathrm{dg}}(R)(C_*, C'_*)_n$ contains $f = \{f_k\}_{k\in\mathbb{Z}}, f_k : C_k \to C'_{k+n}$ that satisfies $\delta_{C'} \circ f_{k+1} = f_k \circ \delta_C$. The differential on $\mathbf{Ch}_{\mathrm{dg}}(\mathbf{k})(C_*, C'_*)$ is given by

$$\delta : \mathbf{Ch}_{\mathrm{dg}}(R)(C_*, C'_*)_n \longrightarrow \mathbf{Ch}_{\mathrm{dg}}(\mathbf{k})(C_*, C'_*)_{n-1} : f \mapsto \delta_{C'} \circ f - (-1)^n f \circ \delta_C$$

The composition in $\mathbf{Ch}_{\mathrm{dg}}$ is the un-shifted composition of complex morphisms. Concretely let $f \in \mathbf{Ch}_{\mathrm{dg}}(R)(C_*, C'_*)_p$, $g \in \mathbf{Ch}_{\mathrm{dg}}(R)(C'_*, C''_*)_q$, then $g \circ f \in \mathbf{Ch}_{\mathrm{dg}}(R)(C_*, C'_*)_{p+q}$ is of the form

$$g \circ f = \{(g \circ f)_k\}_{k\in\mathbb{Z}} = \left\{g^{k+p} \circ f^k : C_k \to C''_{k+p+q}\right\}_k$$

Definition 2.7. For $\mathcal{A}, \mathcal{B}$ two DG categories, a DG functor from $\mathcal{A}$ to $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ induces morphisms of DG $\mathbf{k}$-modules $\mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$ that compatible with composition and unit.

The category of small DG categories $\mathbf{dgcat}_{\mathrm{k}}$ is defined to have objects small DG categories over $\mathbf{k}$ and morphisms to be DG functors.

2.2 Structure of DG Categories

There is a monoidal structure on $\mathbf{dgcat_k}$ inherited from the tensor product of DG $\mathbf{k}$-modules. The tensor product $\mathcal{A} \otimes \mathcal{B}$ has objects $(X, Y), X \in \mathcal{A}, Y \in \mathcal{B}$ with morphisms $\mathcal{A} \otimes \mathcal{B}((X, Y), (X', Y')) = \mathcal{A}(X, X') \otimes \mathcal{B}(Y, Y')$. And clearly that $\mathbf{dgcat_k}$ is further symmetric monoidal category.
Moreover there is an internal Hom in $\mathbf{dgcat_k}$. For $\mathcal{B}, \mathcal{C} \in \mathbf{dgcat_k}$ we set $\mathcal{H}om(\mathcal{B}, \mathcal{C})$ a DG category having objects $F : \mathcal{B} \to \mathcal{C}$ DG functors from $\mathcal{B}$ to $\mathcal{C}$ and morphisms $\mathrm{Hom}(F, G)$ is a DG $\mathbf{k}$-module with the grading $\mathrm{Hom}(F, G)_n = \left\{ \phi = \langle \phi_X \in \mathcal{C}(FX, GX)_n \rangle_{X \in \mathcal{B}} \right\}$ and natural induced differential.

Proposition 2.8. *The functor $\mathcal{H}om : \mathbf{dgcat_k} \times \mathbf{dgcat_k} \longrightarrow \mathbf{dgcat_k}$ is an internal Hom, namely*

$$\mathrm{Hom}_{\mathbf{dgcat}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) = \mathrm{Hom}_{\mathbf{dgcat}}(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C}))$$

We see that $\mathbf{dgcat_k}$ is a closed monoidal category. Moreover, there is a model structure on $\mathbf{dgcat_k}$. Let $\mathcal{A}, \mathcal{B}$ be two DG categories with $F : \mathcal{A} \to \mathcal{B}$ a DG functor. We say F is

- a weak equivalence if $\forall x, y \in \mathcal{A}$, the induced map $\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is an quasi-isomorphism of chain complexes and $H_0(F)$ is an equivalence of categories.
- a fibration if $\forall x, y \in \mathcal{A}$, the induced map $\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a degreewise surjection of complexes and $H_0(F)$ is an isofibration.

Proposition 2.9. *The classes above make $\mathbf{dgcat_k}$ into a model category, called the Tabuada model structure. Moreover, the functor of DG nerve $\mathsf{N}^{\mathrm{dg}}_{\bullet}$ is a right Quillen functor, with respect to Tabuada model structure on $\mathbf{dgcat_k}$ and Joyal model strucutre on $\mathbf{Set}_{\Delta}$.*

Similar to the definition of homotopy category of an ∞-category (Definition 1.35) one can define the homotopy category of a DG category, use the homotopy of chain complexes

Definition 2.10. Let $\mathcal{A}$ be a DG category, define its **homotopy category** $h\mathcal{A}$ to have:

- Objects to be the same as objects of $\mathcal{A}$.
- Morphisms to be $\mathrm{Hom}_{h\mathcal{A}}(X, Y) := H_0\left(\mathrm{Hom}_{\mathcal{A}}(X, Y)_*\right), \forall X, Y \in \mathcal{A}$.

Here H_0 denotes for the 0-th homology group of the chain complex.

Remark 2.11. For $f, f' \in \mathrm{Hom}(X, Y)_0$ in the chain complex $\mathrm{Hom}(X, Y)_*$, we say they are homotopic if there exists $h \in \mathrm{Hom}(X, Y)_1$ such that $\partial(h) = f - f'$. In fact, as is depicted in Example 2.6, we can regard 0-cycle $f \in \mathrm{Hom}(X, Y)_0$ as degree 0 chain map and the homotopy $h \in \mathrm{Hom}(X, Y)_1$ is the usual definition of homotopy of chain complexes. In this case, taking H_0 is exactly quotienting homotopical equivalence relations.

Proposition 2.12. *Let $\mathcal{A}$ be a DG category, then there is an equivalence of categories $h\mathsf{N}^{\mathrm{dg}}_{\bullet}(\mathcal{A}) \cong h\mathcal{A}$, where the LHS is the homotopy category of ∞-category.*

2.3 Dold-Kan correspondence

Recall that for topological space X, we can associate it with a chain complex which has $C_n(X;\mathbb{Z})$ freely generated by $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$ for each degree and the differential is induced by (alternating sum of) face maps. On the other hand, we can define similarly for any simplicial abelian groups (i.e. those in $\mathbf{Set}_\Delta$ valued in $\mathbf{Ab}$) a chain complex. Let $A_\bullet$ be a simplicial abelian group, define

$$C_*(A_\bullet) = \dots \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \longrightarrow 0 \longrightarrow 0 \dots \quad \partial(\sigma) := \sum_{i=0}^{n} (-1)^i d_i^n(\sigma), \forall \sigma \in A_n, n \geqslant 1.$$

One can verify that $\partial^2 = 0$. In general for $S_\bullet \in \mathbf{Set}_\Delta$, denote by $\mathbb{Z}[S_\bullet] \in \mathbf{Ab}_\Delta$ the simplicial abelian group generated by $S_\bullet$, the corresponding chain complex $C_*(\mathbb{Z}[S_\bullet])$ is the chain complex of $S_\bullet$.
Now start from a topological space X, its homology group can be interpreted as following: we first take $\mathrm{Sing}_\bullet(X)$ to get a simplicial set then take $H_n\left(\mathbb{Z}[\mathrm{Sing}_\bullet(X)]\right)$, the result homology group is the same as $H_n(X;\mathbb{Z})$.
Conversely, starting from a chain complex M_*, one can construct topological space. Fix $n \geqslant 0$, let $N_*(\Delta^n;\mathbb{Z})$ be the chain complex with each $N_m(\Delta^n;\mathbb{Z})$ is generated by the collection of non-degenerate (i.e. not the image of any degeneracy map) m-simplex of Δ^n with differential

$$\partial(\sigma) = \sum_{i=0}^{m} (-1)^i \begin{cases} d_i^m(\sigma) & \text{if } d_i^m(\sigma) \text{ non-degenerate} \\ 0 & \text{otherwise} \end{cases} \qquad \forall \sigma \in N_m(\Delta^n;\mathbb{Z})$$

One can verify similarly as above that $N_*(\Delta^n;\mathbb{Z}) \in \mathrm{Ch}(\mathbb{Z})$. Define:

$$K_\bullet(M_*) \in \mathbf{Set}_\Delta \quad \text{with } [n] \mapsto K_n(M_*) := \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}\left(N_*(\Delta^n;\mathbb{Z}), M_*\right)$$

Example 2.13. (Eilenberg-Maclane Spaces) Recall for an abelian group G and $n \geqslant 1$, an Eilenberg-Maclane space $K(G,n)$ is referred to a topological space X such that $\pi_n(X) = G, \pi_i(X) = 0, i \neq n$. For such G, n, we can associate with a complex $G[n]$ that G concentrates in degree n, combining the construction above gives us a simplicial set $K_\bullet(G[n])$. Then taking the geometric realization functor $|K_\bullet(G[n])|$ gives a construction of $K(G,n)$. The simplicial set $K_\bullet(G[n])$ is in fact a Kan complex=∞-groupoid, which is often denoted by $\mathbf{B}^n G$. Let us investigate two cases, when $n = 0, 1$.
When $n = 0$, a chain map $N_*(\Delta^m;\mathbb{Z}) \to G[0]$ is given by

$$\begin{array}{ccccccc} \dots & \longrightarrow & N_1(\Delta^m;\mathbb{Z}) & \xrightarrow{\partial} & N_0(\Delta^m;\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle (g_{ij})_{0\leqslant i<j\leqslant m}=0} & & \downarrow {\scriptstyle (g_0,\dots,g_m)} & & \\ \dots & \longrightarrow & 0 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

The commutativity of the square requires that $g_i - g_j = 0, \forall i < j$, which forces $(g_0, ..., g_m) = (g_0, ..., g_0)$. We can see in this case $K_\bullet(G[0])$ is the constant simplicial abelian group $\underline{G}$.
When $n = 1$, a chain map $N_*(\Delta^m; \mathbb{Z}) \to G[1]$ is given by

$$\begin{array}{ccccccc} ... & \longrightarrow & N_2(\Delta^m;\mathbb{Z}) & \xrightarrow{\partial} & N_1(\Delta^m;\mathbb{Z}) & \xrightarrow{\partial} & N_0(\Delta^m;\mathbb{Z}) \\ & & \downarrow{\scriptstyle (g_{ijk})_{0\leqslant i<j<k\leqslant m}=0} & & \downarrow{\scriptstyle (g_{ij})_{0\leqslant i<j\leqslant m}} & & \downarrow \\ ... & \longrightarrow & 0 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

The commutativity of the square requires that $g_{ij} - g_{ik} + g_{jk} = 0, \forall 0 \leqslant i < j < k \leqslant m$, we can see in this case $K_\bullet(G[1])$ is the classifying simplicial set $\mathbf{B}G$, which is in fact again simplicial abelian group.

The construction of $N_*(\Delta^m; \mathbb{Z})$ can be generalized to any simplicial abelian group as following: For $A_\bullet \in \mathbf{Ab}_\Delta$, consider the subcomplex $D_*(A_\bullet) \subseteq C_*(A_\bullet)$ that each $D_n(A_\bullet)$ is generated by image of $\left\{ s_i^{n-1} : A_{n-1} \to A_n \right\}_{0\leqslant i\leqslant n-1}$. It is not difficult to show that $\partial(D_n(A_\bullet)) \subseteq D_{n-1}(A_\bullet)$ and we take

$$N_*(A_\bullet) := C_*(A_\bullet) \big/ D_*(A_\bullet) \quad \textbf{normalized Moore complex} \text{ of } A_\bullet$$

Theorem 2.14 (Dold-Kan correspondence). *The functor of normalized Moore complex* $N_* : \mathbf{Ab}_\Delta \to \mathrm{Ch}(\mathbb{Z})_{\geqslant 0}$ *is an equivalence category, with inverse functor* $K_\bullet : M_* \mapsto K_\bullet(M_*)$.

The Dold-Kan correspondence 2.14 can be generalized in several forms. An important one for derived algebraic geometry would be that replacing $\mathbf{Ab}_\Delta$ with $\mathbf{Rings}_\Delta$ or $\mathbf{CAlg}_\Delta$, which brings in more structures on one side of Dold-Kan. We are going to see that the other side would be upgraded from connective chain complexes to connective (commutative) DG algebras.

Definition 2.15. Let A be a DG algebra, it is said to be a **commutative differential graded algebra** if the multiplication map μ on A is super-commutative, i.e. $\mu(a \otimes b) = (-1)^{|x|\cdot|y|}\mu(b \otimes a)$. We denote by $\mathrm{dga}_{\mathbf{k}}$ the category of DG $\mathbf{k}$-algebras and by $\mathrm{cdga}_{\mathbf{k}}$ subcategory of commutative DG $\mathbf{k}$-algebras. Also, we use the notation $\mathrm{dga}_{\mathbf{k}}^{\geqslant 0}$ and $\mathrm{cdga}_{\mathbf{k}}^{\geqslant 0}$ for connective ones.

Remark 2.16. Note that both sides of Dold-Kan correspondence are monoidal category, and the two functors are both lax monoidal functor individually. However, they fail to be a monoidal adjunction. Instead, one can have a monoidal Dold-Kan correspondence by considering the monoids in both sides. In particular, the monoids in $\mathbf{Ab}_\Delta$ are $\mathbf{Rings}_\Delta$ and the monoids in $\mathrm{Ch}(\mathbb{Z})_{\geqslant 0}$ are $\mathrm{dga}_{\mathbb{Z}}^{\geqslant 0}$. Then we have the following monoidal Quillen equivalences (with modified Dold-Kan functors):

$$\mathrm{cdga}_{\mathbf{k}}^{\geqslant 0} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \left(\mathbf{CAlg}_{\mathbf{k}}\right)_\Delta^{\mathrm{op}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{cdga}_{\mathbf{k}}^{\geqslant 0} \quad \text{and} \quad \left(\mathbf{CAlg}_{\mathbf{k}}\right)_\Delta^{\mathrm{op}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{cdga}_{\mathbf{k}}^{\geqslant 0}$$

Remark 2.17. As mentioned above that Dold-Kan correspondence is a Quillen adjunction, in fact one can verify that there is a model structure on $\mathrm{dga}_{\mathbf{k}}$, $\mathrm{cdga}_{\mathbf{k}}$, $\mathrm{dga}_{\mathbf{k}}^{\geqslant 0}$, $\mathrm{cdga}_{\mathbf{k}}^{\geqslant 0}$ which is transferred from the one on category of DG $\mathbf{k}$-modules, when $\mathrm{char}(\mathbf{k}) = 0$. Concretely, the weak equivalence on category of chain complexes $\mathrm{Ch}_{\mathbf{k}}^{\geqslant 0}$ is given by quasi-isomorphism the fibration is given by degreewise surjection. In general, we can consider the adjoint pair:

$$\mathrm{Ch}_{\mathbf{k}}^{\geqslant 0} \underset{\mathrm{Fgt}}{\overset{\mathrm{Sym}_{\mathbf{k}}}{\rightleftarrows}} \mathrm{cdga}_{\mathbf{k}}^{\geqslant 0} \quad (\perp)$$

where Fgt is the forgetful functor and $\mathrm{Sym}_{\mathbf{k}}$ is symmetric algebra functor that is left adjoint to Fgt (regardless of $\mathrm{char}(\mathbf{k})$). When $\mathrm{char}(\mathbf{k}) = 0$, then the above adjoint pair is in fact a Quillen adjunction under the above model structure on $\mathrm{cdga}_{\mathbf{k}}^{\geqslant 0}$. However, when $\mathrm{char}(\mathbf{k}) > 0$, it fails to transfer the model structure on $\mathrm{Ch}_{\mathbf{k}}^{\geqslant 0}$ to $\mathrm{cdga}_{\mathbf{k}}^{\geqslant 0}$. Let us consider the following example: the complex $\left[0 \to \mathbf{k} \xrightarrow{\mathrm{id}} \mathbf{k} \to 0\right]$ is nullhomotopic, i.e. weak equivalent to 0 in $\mathrm{Ch}_{\mathbf{k}}^{\geqslant 0}$. Assume the first $\mathbf{k}$ occurs in the complex is in degree $n = 2m, m \geqslant 1$ and the second in degree $n-1$, then $\mathrm{Sym}_{\mathbf{k}}\left(\left[0 \to \mathbf{k} \xrightarrow{\mathrm{id}} \mathbf{k} \to 0\right]\right)$ has underlying algebra $\mathbf{k}[x, y]$ with $\deg(x) = n, \deg(y) = n-1$, moreover, it is cdga with differential $\partial(x) = y, \partial(y) = 0$. If we further assume $\mathrm{char}(\mathbf{k}) = 2$, then $\partial(x^2) = 2xy = 0$, which means $H^{2n}(k[x, y]) \neq 0$, since x^2 is not the boundary of some element. Meanwhile $\mathrm{Sym}_{\mathbf{k}}(0) = \mathbf{k}$ has trivial homology at degree $2n$.
For char p, the corrected replacement of DG algebras is $\mathbb{E}_\infty$-algebras.

References

[DS95] W.G. Dwyer and J. Spalinski. "Homotopy Theories and Model Categories". In: *Handbook of Algebraic Topology*. Elsevier, 1995, pp. 73–126. ISBN: 978-0-444-81779-2. DOI: 10.1016/B978-044481779-2/50003-1.

[Hin15] V. Hinich. *Dwyer-Kan Localization Revisited*. Sept. 2015. DOI: 10.48550/arXiv.1311.4128. arXiv: 1311.4128 [math].

[Kel06] Bernhard Keller. *On Differential Graded Categories*. June 2006. DOI: 10.48550/arXiv.math/0601185. arXiv: math/0601185.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies n° 170. Princeton, N.J: Princeton university press, 2009. ISBN: 978-0-691-14048-3.

[Lur] Jacob Lurie. *Kerodon*. URL: https://kerodon.net/.

[Toë14] Bertrand Toën. "Derived Algebraic Geometry". In: *EMS Surveys in Mathematical Sciences* 1.2 (Nov. 2014), pp. 153–240. ISSN: 2308-2151. DOI: 10.4171/emss/4.