# Working Group: Groups and curvature

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## Introduction

In this lesson, we give a few links between the curvature of a Riemannian manifold and the growth type of its fundamental group.

## 1 Growth type

We start by defining a notion of asymptotic comparison of functions: if $X$ is a subset of $\mathbb{R}$ and $f, g : X \to \mathbb{R}$ are functions, we write $f \preccurlyeq g$ if there exist $a, b > 0$ and $x_0 \in X$ such that for all $x \geqslant x_0$, we have $bx \in X$ and $f(x) \leqslant ag(bx)$.

If both $f \preccurlyeq g$ and $g \preccurlyeq f$ hold, we write $f \asymp g$ and say that $f$ and $g$ are *asymptotically equivalent*.

Let $G$ be a finitely generated group with generating set $S = \{s_1, \ldots, s_m\}$. This defines a length function on $G$:

$$l_S : g \mapsto \min \left\{ \sum_i |a_i| \mid g = \prod_i s_i^{a_i} \right\}$$

and a left-invariant distance, also called the *word metric*

$$d_S : (g, h) \mapsto l_S(g^{-1}h)$$

which can be seen as the edge distance on the Cayley graph $Cay(G, S)$.
If $T$ is another finite generating set of $G$, then it is easy to see that there exists $C \geqslant 1$ such that for all $g \in G$,

$$\frac{1}{C} l_S(g) \leqslant l_T(g) \leqslant C l_S(g).$$

Thus, if we set for all $r \geqslant 0$, $\varphi_G^S(r) = |\{g \in G \mid l_S(g) \leqslant r\}|$, we have $\varphi_G^S \asymp \varphi_G^T$ and it is possible to talk about the asymptotic growth function $\varphi_G$ of a finitely generated group $G$, well defined up to the asymptotic equivalence relation $\asymp$.

**Definition 1.** Let $G$ be a finitely generated group, $S$ be a generating set. $G$ is said to have

- *Polynomial growth* of degree $d \geqslant 0$ if there exists $C \geqslant 0$ such that for all $r \geqslant 0$, $\varphi_G^S(r) \leqslant Cr^d$.
- *Exponential growth* if there exists $C > 1$ such that for all $r \geqslant 0$, $\varphi_G^S(r) \geqslant C^r$.
- $G$ is said to have *intermediate growth* otherwise.

**Example 1.**

(i) $\mathbb{Z}^d$ has polynomial growth of degree $d$.

(ii) If $G$ is the free group generated by $S$, then $\varphi_G^S(r) = 1 + m\frac{(2m-1)^r-1}{m-1}$.

(iii) If $m \geqslant 1$ and $G$ is a free abelian group on the set $S = \{s_1, \dots, s_m\}$, then $\varphi_G^S(r) = \sum_{i=0}^m 2^i \binom{m}{i}\binom{r}{i} = O(r^m)$.

(iv) The Grigorchuk groups have intermediate growth between $\exp(\sqrt{r})$ and $\exp(r^\alpha)$ for $\alpha < 1$, see [Gri83].

The notion of growth has many applications in group theory, for instance we have the following results.

**Theorem 1.** *Let $G$ be a finitely generated group, $H$ be a finitely generated subgroup of $G$.*

*(i) Then $\varphi_H \preccurlyeq \varphi_G$.*

*(ii) If $H$ has finite index, then $\varphi_H \asymp \varphi_G$.*

*(iii) If $H$ is normal in $G$, then $\varphi_{G/H} \preccurlyeq \varphi_G$.*

*(iv) If $H$ is finite normal in $G$, then $\varphi_{G/H} \asymp \varphi_G$.*

*Proof.* (i) If $T$ is a finite generating set of $H$ and $S$ is a finite generating set of $G$ containing $T$, then the Cayley graph $Cay(H,T)$ is a subgraph of $Cay(G,S)$. In particular, for every $h \in H$, $l_S(h) \leqslant l_T(h)$ considered as distances on the same connected graph $Cay(G,S)$. Thus, a ball of radius $r$ for $d_T$ is contained in the ball of radius $r$ for $d_S$ with the same center, hence the result.

(ii) This proof requires the introduction of a few notions of coarse geometry, we refer to [DK18] for a detailed explanation of the subject. A metric space $(X, d)$ is said to be *geodesic* if each pair of points $x, y \in X$ is connected by a geodesic, where the geodesic need not be unique, it is *proper* if every closed, bounded subset is compact. An action of a finitely generated group $G$ on a a metric space $(X, d)$ is *geometric* if $G$ acts cocompactly by isometry on $X$ and if this action is properly discontinuous. Finally, a map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is a *quasiisometry* if there exists $A \geqslant 1, B, C \geqslant 0$ such that the following conditions hold:

(a) For all $x, y \in X$, $A^{-1} d_X(x, y) - B \leqslant d_Y(f(x), f(y)) \leqslant A d_X(x, y) + B$.

(b) For all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leqslant C$.

We will use the following.

**Theorem 2.** *(Milnor-Schwarz)[Šva55] Let $(X, d)$ be a proper geodesic metric space and let $G$ be a group acting geometrically on $X$. Then:*

*(a) The group $G$ is finitely generated.*

*(b) For any word metric $d_G$ on $G$ and any point $x \in X$, the orbit map $G \to X$ given by $g \mapsto gx$ is a quasiisometry.*

A finitely generated group $G$ equipped with the word metric is clearly a proper geodesic metric space on which $H$ acts geometrically when it has finite index in $G$, thus there exists a quasiisometry $f : H \to G$ between them. In particular, the definition of a quasiisometry implies the existence of a *coarse inverse* of $f$, that is a quasiisometry $g : G \to H$ such that there exists $C \geqslant 0$ verifying for all $x \in H$ and $y \in G$,

$$d_H(g \circ f(x), x) \leqslant C \quad \text{and} \quad d_G(f \circ g(y), y) \leqslant C.$$

We will now show that two quasiisometric groups $G$ and $H$ verify $\varphi_G \asymp \varphi_H$.

Let $A \geqslant 1, B \geqslant 0$ be such that for all $x, x' \in H$, $y, y' \in G$

$$A^{-1} d_H(x, x') - B \leqslant d_G(f(x), f(x')) \leqslant A d_H(x, x') + B$$

$$A^{-1} d_G(y, y') - B \leqslant d_H(g(y), g(y')) \leqslant A d_G(y, y') + B.$$

In particular, if $S$ is a finite generating set of $G$ and $T$ is a finite generating set of $H$, they induce metric spaces $(G, d_G)$ and $(H, d_H)$ and we have for all $x, y \in H$

$$d_G(f(x), f(y)) \leqslant (A + B) d_H(x, y)$$

that is $f$, and $g$ by the same argument, is $(L = A + B)$-Lipschitz. Fix $x_0 \in H$, $y_0 \in G$ and let $D = \max\left(d_G\left(f\left(x_0\right), y_0\right), d_H\left(x_0, g\left(y_0\right)\right)\right)$. Then for each $R > 0$,

$$f\left(\bar{B}\left(x_0, R\right)\right) \subset \bar{B}\left(y_0, LR + D\right), \quad g\left(\bar{B}\left(y_0, R\right)\right) \subset \bar{B}\left(x_0, LR + D\right)$$

while $f(x) = f\left(x'\right)$ implies $d\left(x, x'\right) \leqslant AB$. The same applies to the map $g$. Since the spaces $H$ and $G$ are uniformly discrete, for both maps $f, g$ the cardinality of the preimage of a point is smaller than $m$, where $m$ is an upper bound for the cardinality of closed balls of radius $AB$ in $H$ and $G$. It follows that

$$|\bar{B}\left(x_0, R\right)| \leqslant m|\bar{B}\left(y_0, LR + D\right)|$$

and

$$|\bar{B}\left(y_0, R\right)| \leqslant m|\bar{B}\left(x_0, LR + D\right)|,$$

that is $\varphi_H \asymp \varphi_G$ and concludes the proof of (ii).

(iii) Let $S$ be a finite generating set of $G$ and $\bar{S}$ be the corresponding finite generating set of $G/H$. Then the canonical surjection $G \to G/H$ maps the ball of center $e$ and radius $r$ onto the ball of center $e$ and radius $r$ in $G/H$.

(iv) The argument is the same as for (ii).

□

**Remark 1.** Take the Heisenberg group $H$, and the subgroup $H_{\mathbb{Z}}$ of $H$ obtained by taking integer parameters. For instance, these results show that $H_{\mathbb{Z}}$ does not contain the free group on two generator $\mathbb{F}_2$ as a subgroup.

As a final application of growth to group theory, we give the following result proved in [Gro81].

**Theorem 3.** *(Gromov) A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

# 2 Riemannian geometry

We now remind a few notions which will be useful to state and prove results linking the growth of balls in a manifold to the growth of its fundamental group.

## 2.1 Geodesics

Let $(X, d)$ be a metric space. A *path* in $X$ is a continuous map $\gamma : [a, b] \to X$, it is said to *join* points $x, y \in X$ if $\gamma(a) = x$ and $\gamma(b) = y$. Given a path $\gamma$ in $X$, one defines the *length* of $\gamma$ as follows:

$$\text{length}(\gamma) = \sup_{\substack{n\in\mathbb{N} \\ a=t_0<t_1<\cdots<t_n=b}} \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

The path $\gamma$ is said to be *rectifiable* if $\text{length}(\gamma) < +\infty$. We can define a pseudometric $d_\ell$ on $X$ by

$$d_\ell(x, y) = \inf \{\text{length}(\gamma) \,|\, \gamma \text{ rectifiable joins } x \text{ and } y\}.$$

A *geodesic* in a metric space $(X, d)$ is a continuous curve $\gamma\colon I \to X$ such that for all $t \in I$, there exists an open neighborhood $J$ of $t$ verifying for all $t_1, t_2 \in J$

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|.$$

**Remark 2.** When the considered metric on $X$ is $d_\ell$, the notion of (smooth) geodesic coincides with the definition of geodesic in Riemannian manifolds as a parametrized curve with zero covariant derivative.

The main result about geodesics is that they exist locally in any Riemannian manifold.

**Theorem 4.** *(Local existence and uniqueness of geodesics) Let $(M, g)$ be a Riemannian manifold, for every $m_0 \in M$, there exists an open set $m \in U \subseteq M$ and $\varepsilon > 0$ such that for $m \in U$ and $v \in T_mM$ with $\|v\| < \varepsilon$, there is a unique smooth geodesic $\gamma\colon ]-1, 1[ \to M$ such that $\gamma(0) = m$ and $\gamma'(0) = v$.*

**Definition 2.** A Riemannian manifold is said to be *geodesically complete* if any geodesic of $M$ can be extended to a geodesic defined on $\mathbb{R}$.

This is the case for instance when each closed bounded subset of $M$ is compact, other conditions equivalent are given by the Hopf-Rinow Theorem, see [HR31].

## 2.2 Cut-locus

Let $(M, g)$ be a complete Riemannian manifold. Let $p \in M$ and $\gamma$ be a geodesic in $M$ with $\gamma(0) = p$. For $t > 0$ small enough $\gamma_{|[0,t]}$ is length minimizing between $\gamma(0)$ and $\gamma(t)$. For a general $t > 0$, it may happen that there exists $t_0$ such that $\gamma_{|[0,t]}$ is no longer length minimizing between $\gamma(0)$ and $\gamma(t)$ for all $t > t_0$, this motivates the following definition.

**Definition 3.** Let $(M, g)$ be a complete Riemannian manifold, $p \in M$ a point, and $\gamma : [0, \infty[ \to M$ a geodesic with $\gamma(0) = p$. If

$$t_0 := \sup\{t \mid \gamma([0,t]) \text{ is a minimizing geodesic } \} < +\infty$$

then we will call $\gamma(t_0)$ the *cut point* of $p$ along $\gamma$.

The *cut locus* of $p$ in $M$ is defined to be the set $\mathrm{Cut}(p)$ of all cut points of $p$ along geodesics that start from $p$.

Note that if $M$ is compact, then $\mathrm{Cut}(p) \neq \varnothing$ for all $p \in M$. Also, the cut-locus of a point has measure zero, see [Cha95, Proposition III.3.1].

**Example 2.**

(i) On $\mathbb{R}^m$ and $\mathbb{H}^m$ (endowed with the canonical metrics), there exists only one minimizing geodesic joining any two given points. So $\mathrm{Cut}(p) = \varnothing$ for all $p$.

(ii) For $\mathbb{S}^m$ with the metric of constant curvature, $\mathrm{Cut}(p)$ is the antipodal point of $p$.

# 3 Growth comparison theorems: manifolds and their fundamental group

A key ingredient in comparing growth of balls in a manifold $M$ and in its fundamental group is the action of $\pi_1(M, x_0)$ on the universal cover $\tilde{M}$. If the base space is a Riemannian manifold, then the universal cover can be equipped with a Riemannian structure such that $\pi_1(M, x_0)$ acts by isometries.

**Definition 4.** Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds. A map $p : N \to M$ is a *Riemannian cover* if it is a smooth covering map and a local isometry.

We will mainly be focusing on the universal cover of Riemannian manifolds, the following Theorem says that it can canonically be equipped with a Riemannian structure compatible with the metric of the base space.

**Theorem 5.** *Let $p : N \to M$ be a smooth covering map. For any Riemannian metric $g$ on $M$, there exists a unique metric $h$ on $N$ such that $p$ is a Riemannian covering map.*

*Proof.* If such a metric $h$ exists, it must satisfy for all $n \in N$ and $X, Y \in T_nN$:

$$h_n(X, Y) = g_{p(n)}(T_{p(n)}p(X), T_{p(n)}p(Y)).$$

Conversely, since $T_n p$ is a vector space isomorphism for all $n$, this defines a scalar product on each tangent space of $N$. Because $p$ is a local diffeomorphism, $h$ and $g$ have the same expression respectively in a local chart $(U, \phi)$ around a given point $n \in N$ and in $(p(U), \phi \circ p^{-1})$ around $p(n)$. This shows that $p$ is smooth and that it is a Riemannian covering map. $\square$

Recall $\pi_1(M)$ acts on the universal covering $p : M \to \tilde{M}$ in the following way: if $y_0 \in M$ and $x_0$ belongs to the fiber of $y_0$, then for any $z \in \tilde{M}$, take a path $\alpha$ from $x_0$ to $z$ (unique up to homotopy in the universal cover) and set $\beta = p \circ \alpha$. Then, for any $\gamma \in M$, let $\xi = \gamma \circ \beta$ be a path from $y_0$ to $p(z)$ and $\tilde{\xi}$ be the unique lift of $\xi$ starting at $x_0$. The action of $\gamma$ on $z$ is defined as

$$\gamma \cdot z = \tilde{\xi}(1).$$

Let $(M, g)$ be a connected Riemannian manifold with universal covering $p : \tilde{M} \to M$. The path pseudo-metric $d_\ell$ on $M$ induces $\tilde{d_\ell}$ on $\tilde{M}$ defined by

$$\tilde{d_\ell}(x, y) = \inf\{\text{length}(p \circ \gamma) \mid \gamma \text{ joins } x \text{ and } y\}.$$

We claim that the fundamental group of $M$ acts isometrically on the universal cover $(\tilde{M}, \tilde{d_\ell})$. To see this, take $g \in \pi_1(M, x_0)$ and $a, b \in \tilde{M}$. If $\gamma$ is a path in $\tilde{M}$ joining $a$ and $b$, then $g \cdot \gamma$ joins $g \cdot a$ and $g \cdot b$. Denoting $\Gamma(a, b)$ the set of paths from $a$ to $b$ in $\tilde{M}$, we have a bijection

$$\begin{aligned} \Gamma(a, b) &\to \Gamma(g \cdot a, g \cdot b) \\ \gamma &\mapsto g \cdot \gamma. \end{aligned}$$

If $p : M \to \tilde{M}$ is the associated covering, we have for every path $\gamma$ in $\tilde{M}$ and $g \in \pi_1(M, x_0)$

$$p(\gamma) = p(g \cdot \gamma)$$

which means that $p\left(\Gamma(a, b)\right) = p\left(\Gamma(g \cdot a, g \cdot b)\right)$. Finally,

$$\tilde{d_\ell}(a, b) = \inf_{\gamma \in p(\Gamma(a,b))} \text{length}(\gamma) = \inf_{\gamma \in p(\Gamma(g \cdot a, g \cdot b))} \text{length}(\gamma) = \tilde{d_\ell}(g \cdot a, g \cdot b).$$

For every $n$-dimensional Riemannian manifold $(M, g)$ one defines the volume element denoted dV: given $n$ vectors $(v_1, \ldots, v_n)$ in $T_pM$, $dV(v_1 \wedge \cdots \wedge v_n)$ is the volume of the parallelepiped in $T_pM$ spanned by these vectors. The volume of a subset $A$ of $M$ is then

$$\text{Vol}(A) = \int_A dV.$$

**Theorem 6.** *Let $(M, g)$ be a Riemannian manifold and let $G$ be a finitely generated subgroup of $\pi_1(M, x_0)$. Then for all $a$ in the universal Riemannian cover $\tilde{M}$ of $M$, $\varphi_G(r) \preccurlyeq \mathrm{Vol}(B(a, r))$.*

*Proof.* Take $a \in \tilde{M}$. From the definition of covering maps, we can find $r > 0$ such that the balls $B(\gamma(a), r)$ are pairwise disjoint. Take a finite system $S$ of generators of the subgroup $H$ of $\pi_1(M, x_0)$ we consider, and set

$$L = \max d\left(a, \gamma_i(a)\right), \quad \gamma_i \in S$$

Now, if $\gamma \in H$ can be represented as a word of length not greater than $s$ with respect to the $\gamma_i$, we have

$$d(a, \gamma(a)) \leq Ls.$$

Taking all such $\gamma$, we obtain $\varphi_S(s)$ disjoint balls $B(\gamma(a), r)$, such that

$$B(\gamma(a), r) \subseteq B(a, Ls + r)$$

Therefore

$$\varphi_G^S(s) \leq \frac{\mathrm{Vol}(B(a, Ls + r))}{\mathrm{Vol}(B(a, r))}.$$

□

**Proposition 1.** *Let $(M, g)$ be a compact connected Riemannian manifold. There exists a compact subset $K \subseteq \tilde{M}$ such that $(\gamma(K))_{\gamma \in \pi_1(M)}$ is a locally finite covering of $\tilde{M}$.*

*Proof.* Since $M$ is compact, we can choose a finite covering of $M$ by open sets $(U_i)_i$ on each of which the covering map is a homeomorphism. Up to restricting these open sets, by local compactness of $M$, each $U_i$ can be taken of compact closure, this yields a finite compact covering of $M$ which lifts to a compact $K$ of $\tilde{M}$. Recall that the monodromy action of $\pi_1(M, x_0)$ on $\tilde{M}$ is transitive on the fibers of the covering. Since $K$ meets each fiber, the translated $\gamma(K)$ of $K$ cover $\tilde{M}$ when $\gamma$ runs over $\pi_1(M, x_0)$.

Take the universal Riemannian covering $(\tilde{M}, \tilde{g})$ of $(M, g)$ (*i.e.* a smooth covering that is also a local isometry). Let $d$ be the distance which is given by $\tilde{g}$. As a consequence of the Lebesgue property for the compact $K$, there exists some $r > 0$ such that, for any ball $B$ of radius $r$ whose center lies in $K$, the balls $\gamma(B)$ are pairwise disjoint when $\gamma$ ranges over $\pi_1(M)$. Let us now show that for any $x$ in $\tilde{M}$, the ball $B\left(x, \frac{r}{2}\right)$ only meets a finite number of $\gamma(K)$. Since the $\gamma$ 's are isometries, we can suppose that $x$ lies in $K$. Suppose there exists a sequence $\gamma_n$ of distinct elements of $\Gamma$, and a sequence $y_n$ of points of $K$, such that for any $n$

$$\gamma_n\left(y_n\right) \in B\left(x, \frac{r}{2}\right).$$

Up to extracting a subsequence, we can suppose that $y_n$ converges in $K$ to $y \in K$. Then, since the $\gamma_n$ are isometries, $\gamma_n(y)$ belongs to $B(x,r)$ for all $n$ big enough which is in contradiction with the fact that balls $\gamma(B)$ are disjoint.

□

A direct consequence of Proposition 1 is the following: for all $D > 0$, the set

$$S(D) = \{\gamma \in \pi_1(M), d(K, \gamma(K)) < D\}$$

is finite.

**Lemma 7.** *Take $D > \delta = diam(M)$. Take $a \in K$, and $\gamma \in \pi_1(M)$ such that, for some integer $s$,*

$$d(a, \gamma(K)) \leq (D - \delta)s + \delta.$$

*Then $\gamma$ can be written as the product of $s$ elements of $S(D)$.*

*Proof.* Take $y \in \gamma(K)$, a minimizing geodesic $c$ from $a$ to $y$, and points $y_1, y_2, \cdots, y_{s+1}$ such that

$$d(a, y_1) < \delta \quad \text{and} \quad d(y_i, y_{i+1}) \leq (D - \delta) \quad \text{for} \quad 1 \leq i \leq s.$$

Any $y_i$ can be written as $\gamma_i(x_i)$, for some $\gamma_i$ in $\pi_1(M)$ and some $x_i$ in $K$, and we can take $\gamma_1 = Id$ and $\gamma_{s+1} = \gamma$. Then

$$\gamma = \left(\gamma_1^{-1}\gamma_2\right)\left(\gamma_2^{-1}\gamma_3\right)\ldots\left(\gamma_s^{-1}\gamma_{s+1}\right).$$

On the other hand

$$d\left(x_i, \gamma_{i-1}^{-1}\left(\gamma_i\left(x_i\right)\right) = d\left(\gamma_{i-1}\left(x_i\right), y_i\right)\right.$$

is smaller than

$$d\left(\gamma_{i-1}\left(x_i\right), \gamma_{i-1}\left(x_{i-1}\right)\right) + d\left(\gamma_{i-1}\left(x_{i-1}\right), y_i\right)).$$

But this is just $d(x_{i-1}, x_i) + d(y_{i-1}, y_i)$, which is smaller than $D$, so that $\gamma_{i-1}^{-1}\gamma_i$ is in $S$. □

**Theorem 8.** *If $(M, g)$ is a compact Riemannian manifold, then*

$$\mathrm{Vol}(B(a, r)) \preccurlyeq \varphi_{\pi_1(M)}(r).$$

*Proof.* Take a system $S$ of generators as Lemma 7. This Lemma says that the ball

$$B(a, (D - \delta)s + \delta)$$

is covered by $\varphi_G^S(s)$ compact sets $\gamma(K)$, so that

$$\mathrm{Vol}\left(B(a, (D - \delta)s + \delta\right) \leq \varphi_G^S(s)\,\mathrm{Vol}(K).$$

□

# 4 Curvature and growth of the fundamental group

## 4.1 Curvature

We begin by introducing various notions linked to the curvature of a Riemannian manifold. Let $\pi : TM \to M$ be the tangent bundle of the manifold $M$ and denote by $\Gamma(TM)$ the set of smooth section of $TM$.

**Definition 5.** A *connection* on $TM$ is a continuous map

$$\begin{aligned} \nabla \colon \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, S) &\mapsto \nabla_X S \end{aligned}$$

verifying for all$f \in \mathcal{C}^\infty(M)$ and all $X, S \in \Gamma(TM)$:

(i) $\nabla_X(fS) = f\nabla_X S + X(f)S$

(ii) $\nabla_{fX} S = f\nabla_X S$.

This definition extends to the case of a general vector bundle $\pi : E \to M$ but this will not be needed here.

**Definition 6.** A connection $\nabla$ on $TM$ is said to be *torsion-free* if for all $X, Y \in \Gamma(TM)$, $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$, where $[X, Y] = L_X L_Y - L_Y L_X$ is the Lie bracket on $\Gamma(TM)$.

**Theorem 9.** *On any Riemannian manifold* $(M, g)$*, there exists a unique torsion-free connection* $\nabla$ *called the* Levi-Civita connection *which is consistent with the metric,* i.e. *such that for all* $X, Y, Z \in \Gamma(TM)$*,*

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

For the proof, see for instance [GHL90, Th. 2.51]

**Example 3.** For instance, on $\mathbb{R}^n$, the Levi-Civita connection is $\nabla_X Y = dY(X)$.

Connections can be used to define various notions of curvature on a manifold.

**Definition 7.** Let $(M, g)$ be a Riemannian manifold and $\nabla$ be a connection on $TM$. The *curvature tensor* on $M$ is

$$R(X, Y)S = \nabla_{[X,Y]}S + \nabla_X \nabla_Y S - \nabla_Y \nabla_X S.$$

By considering subplanes of a tangent space to a manifold, we can define a notion of curvature analogous to the two-dimensional case of surfaces.

**Definition 8.** Let $(M, g)$ be a Riemannian manifold, $m \in M$ and $\{x, y\}$ be two independent vectors spanning a plane $P \subseteq T_m M$. The *sectional curvature* of $M$ at $m$ in the plane $P$ is

$$K(x, y) = \frac{R(x, y, x, y)}{|x \wedge y|}.$$

**Definition 9.** Let $(M, g)$ be a Riemannian manifold of dimension $n \geqslant 1$, $m \in M$, $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_m M$ and $\{x, y\}$ be two independent vectors spanning a plane $P \subseteq T_m M$. The *Ricci curvature* of $M$ at $m$ in the plane $P$ is defined as

$$\mathrm{Ric}_m(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} R(x, e_i, y, e_i).$$

## 4.2 Comparing growth and curvature

Let $n$ be a positive integer. There is a canonical metric $g_0$ called the *flat metric* on $\mathbb{R}^n$ given by $g_0 = \sum_{i=1}^n dx_i^2$. Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$, the Riemannian metric induced on $\mathbb{S}^n$ by $g_0$ is called the *round metric* which we denote by $d_{\mathrm{round}}$. Finally, on $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$, we define a metric on $\mathbb{H}^n$ by $g_{\mathrm{hyp}} = x_n^{-1} \left(\sum_{i=1}^n dx_i^2\right)$.

For any constant $a \in \mathbb{R}$, they yield simply connected Riemannian manifolds of constant sectional curvature $a$, namely

- $(\mathbb{S}^n, a^{-1} g_{\mathrm{round}})$ if $a > 0$;
- $(\mathbb{R}^n, g_0)$ if $a = 0$;
- $(\mathbb{H}^n, -a^{-1} g_{\mathrm{hyp}})$ if $a < 0$.

They are the canonical models of such manifolds of constant curvature in the following sense.

**Theorem 10.** *(Killing-Hopf) Let $(M, g)$ be a complete Riemannian manifold of constant sectional curvature $a$, then the Riemannian universal cover of $(M, g)$ is one of the above depending on the sign of $a$.*

Denote by $V^a(r)$ the volume of the ball of radius $r > 0$ in the corresponding manifold of constant curvature $a$. We have

$$V^a(r) \asymp \begin{cases} r^n \text{ if } a \geqslant 0 \\ e^r \text{ if } a < 0. \end{cases}$$

**Lemma 11.** *(Bishop-Gunther) Let $(M, g)$ be a complete Riemannian manifold of dimension $n$ and $m \in M$, $r > 0$ be such that $B(m, r) \cap \mathrm{Cut}(m) = \varnothing$.*

- *If there exists $a \in \mathbb{R}$ such that* $\mathrm{Ric} \geqslant (n-1)ag$*, then* $\mathrm{Vol}(B(m,r)) \leqslant V^a(r)$.
- *If there exists $b \in \mathbb{R}$ such that $K \leqslant b$, then* $\mathrm{Vol}(B(m,r)) \geqslant V^b(r)$.

*Proof.* This proof uses a few notions of Riemannian geometry we did not introduce here, for a more self contained explanation, see [GHL90]. Take $u \in T_mM$ and a geodesic $c(t) = \exp_m(tu)$ from $m$, and an orthonormal basis $\{u, e_2, \dots, e_n\}$ of the tangent space of $M$ at $m$. Take also the parallel vector fields $E_i$, $2 \le i \le n$ along $c$ such that $E_i(0) = e_i(0)$. Let $[0, \rho(u)]$ be the maximal interval such that $c$ is minimal. Suppose that

$$0 \le r \le \rho(u).$$

For such an $r$, there exists a unique Jacobi field $Y_i^r$ such that

$$Y_i^r(0) = 0 \quad \text{and} \quad Y_i^r(r) = E_i(r).$$

Indeed, since $T_{ru}\exp_m$ is an isomorphism from $T_mM$ onto the tangent space $T_{c(r)}M$, this Jacobi field is given by

$$Y_i^r(t) = T_{tu}\exp_m(tv)$$

where $v$ is the unique tangent vector at $m$ such that

$$T_{ru}\exp_m(rv) = E_i(r).$$

Now,

$$J(u,t) = C_r t^{1-n} \det\left(Y_2^r(t), \cdots, Y_n^r(t)\right)$$
$$\text{where} \quad C_r^{-1} = \det\left(Y_2'^r(0), \cdots, Y_n'^r(0)\right)$$

For a given $u$, set $f(t) = J(u,t)$. The *index form of energy* for a vector field $X$ on $M$ is

$$I(X,X) = \frac{d^2}{dt^2}|_{t=0} E(c_t)$$

where $c$ is any smooth curve such that $\frac{d}{dt}|_{t=0} c_t = X$ and the energy of a curve $c$ defined on $[0,t]$ is

$$E(c) = \int_0^t |c'(u)| du.$$

**Lemma 12.** *We have*

$$\frac{f'(r)}{f(r)} = \sum_{i=2}^{n} I\left(Y_i^r, Y_i^r\right) - \frac{(n-1)}{r}$$

*Proof.* First remark that

$$|\det(Y_2^r, \dots Y_n^r)| = \left(\det g\left(Y_i^r, Y_j^r\right)\right)^{1/2}$$

In other words, denoting this last determinant by $D(t)$, we have

$$\frac{f'(t)}{f(t)} = \frac{D'(t)}{2D(t)} - \frac{n-1}{t}$$

For $t = r$, the matrix $\left[g\left(Y_i^r, Y_j^r\right)\right]$ is just the unit matrix,

$$D'(r) = 2\sum_{i=2}^{n} g\left((Y_i^r)', Y_i^r\right)$$

On the other hand the second variation formula when applied to a Jacobi field $Y$, gives

$$I(Y,Y) = \int_0^r \left(|Y'|^2 - R(Y, c', Y, c')\right) ds = \left[g(Y, Y')\right]_0^r$$

The claimed formula is now straightforward. □

**Lemma 13.** *If $c : [a,b] \to M$ is a minimizing geodesic, $Y$ is a Jacobi field and $X$ is a vector field along $c$ with the same values as $Y$ at the ends, then $I(X,X) \geq I(Y,Y)$.*

*Proof.* Proof of the lemma. Since $X - Y$ vanishes at the ends, we have

$$I(X - Y, X - Y) \geq 0$$

because $c$ is minimizing. On the other hand we have

$$I(Y,Y) = \left[g(Y', Y)\right]_a^b \quad \text{and} \quad I(X,Y) = \left[g(Y', X)\right]_a^b.$$

Therefore $I(X - Y, X - Y) = I(X,X) - I(Y,Y)$ and the result follows. □

End of the proof of the theorem:

Proof of i): we shall apply Lemma 13 to $Y_i^r$ and to the vector field $X_i^r$ given by

$$X_i^r(t) = \frac{s(t)}{s(r)} E_i(t),$$

where

$$\begin{cases} s(t) = \sin\sqrt{a}t & \text{if} \quad a > 0 \\ s(t) = t & \text{if} \quad a = 0 \\ s(t) = \sinh\sqrt{-a}t & \text{if} \quad a < 0. \end{cases}$$

Lemma 13 gives

$$\sum_{i=2}^{n} I\left(Y_i^r, Y_i^r\right) \leq \sum_{i=2}^{n} I\left(X_i^r, X_i^r\right).$$

The right member of this inequality is just

$$\int_0^r \left(\frac{s(t)}{s(r)}\right)^2 \left((n-1)a - \mathrm{Ric}\left(c', c'\right)\right) ds + \sum_{i=2}^{n} g\left(X_i^r, (X_i^r)'\right)(r)$$

The assumption made on the curvature yields that the integral is negative. Then, using lemma 12 and the definition of $X_i^r$, we see that

$$\begin{cases} \frac{f'(r)}{f(r)} \leq (n-1)\left(\sqrt{a}\,\mathrm{cotan}\,\sqrt{a}r - \frac{1}{r}\right) & \text{if} \quad a > 0 \\ \frac{f'(r)}{f(r)} \leq 0 & \text{if} \quad a = 0 \\ \frac{f'(r)}{f(r)} \leq (n-1)\left(\sqrt{-a}\,\mathrm{cotanh}\,\sqrt{-a}r - \frac{1}{r}\right) & \text{if} \quad a < 0. \end{cases}$$

In any case, if $f_a(r)$ denotes the function $J(u,r)$ for the "model space" with constant curvature $a$ (recall that $J$ does not depend on $u$ in that case), we have

$$\frac{f'(r)}{f(r)} \leq \frac{f_a'(r)}{f_a(r)}$$

By integrating, we get $f(r) \leq f_a(r)$. The claimed inequality follows from a further integration, the fact that:

$$\mathrm{Vol}(M,g) = \int_{\mathbb{S}^{n-1}} \int_0^{\rho(u)} J(u,t) t^{n-1} dt du$$

Proof of ii): denoting by $Y$ one of the Jacobi fields $Y_i^r$, we have

$$\begin{aligned} g\left(Y(r), Y'(r)\right) &= \int_0^r \left(g\left(Y', Y'\right) - R\left(Y, c', Y, c'\right)\right) ds \\ &\geq \int_0^r \left(g\left(Y', Y'\right) - b g(Y,Y)\right) ds \end{aligned}$$

Write

$$Y(t) = \sum_{i=2}^{n} y^i(t) E_i(t)$$

On the simply connected manifold with constant curvature $b$, take a geodesic $\tilde{c}$ of length $r$, and define vector fields $\tilde{E}_i$ along $\tilde{c}$ in the same way as the vectors $E_i$. Set

$$\tilde{Y}(t) = \sum_{i=2}^{n} y^i(t) \tilde{E}_i(t),$$

then
$$\int_0^r \left( \left| \tilde{Y}' \right|^2 - b |\tilde{Y}|^2 \right) dt = \int_0^r \left( |Y'|^2 - b |Y|^2 \right) dt = I(\tilde{Y}, \tilde{Y}).$$

Lemma 13, when applied to the simply connected manifold with constant curvature $b$, gives
$$I\left(\tilde{Y}_i^r, \tilde{Y}_i^r\right) \geq I\left(\tilde{X}_i^r, \tilde{X}_i^r\right)$$

where $\tilde{X}_i^r(t) = \frac{s(t)}{s(r)} \tilde{E}_i(t)$ is the Jacobi field which takes at the ends of $\tilde{c}$ the same values as $\tilde{Y}_i^r$. Using lemma 12, we see that

$$\frac{f'(r)}{f(r)} \geq \frac{f_b'(r)}{f_b(r)}$$

and the claim follows by integration. □

We are now ready to give the announced links between curvature and growth.

**Remark 3.** For a Riemannian manifold $(M, g)$ with constant sectional curvature $a \in \mathbb{R}$, $\mathrm{Ric} = (n-1)ag$.

**Theorem 14.** *(Milnor-Wolf, cf [Mil68], [Wol68]) Let $(M, g)$ be a complete Riemannian manifold with nonnegative Ricci curvature. Then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree at most* $\dim(M)$.

*Proof.* This is a consequence of Theorem 6 and Bishop-Gunther's Lemma 11 (i). □

Note that the same property holds for $\pi_1(M)$ if $M$ is compact since the fundamental group of a compact manifold is finitely generated (see [Mye35]).

**Example 4.** It can be proved (cf. [Wol68]) that the Heisenberg group with integer coefficients $H_{\mathbb{Z}}$ has polynomial growth of degree 4. Therefore, the compact manifold $H/H_{\mathbb{Z}}$ carries no metric with nonnegative Ricci curvature.

**Theorem 15.** *(Milnor, cf. [Mil68]) If $(M, g)$ is a compact manifold with strictly negative sectional curvature, then $\pi_1(M)$ has exponential growth.*

*Proof.* This is a consequence of Theorem 8 and Lemma 11 (ii). □

As a consequence of the above Theorem, we obtain the following result.

**Corollary 16.** *There is no metric on the torus $\mathbb{T}^n$ with strictly negative curvature.*

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