Groups and curvature

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*A star * denotes a section or a proof that was omitted during the oral presentation.*

1 End of the proof of Milnor-Wolf

Let M be a compact connected Riemannian manifold. Recall that our goal is to establish a link between the growth of $\pi_1(M)$ and the curvature of M. To do this, we first relate the growth of $\pi_1(M)$ to that of the volume of a ball in the universal covering $\tilde{M}$. Last session, the first part of this relation was proved ; we showed that the growth function ϕ of $\pi_1(M)$ satisfy

$$\phi(n) \preccurlyeq \operatorname{vol}(B_{\tilde{M}}(a, n))$$

for $a \in \tilde{M}$ (recall that $f \preccurlyeq g$ means that there exists a constant C such that $f(n) \leq Cg(Cn)$ for every $n \in \mathbb{N}$). Now we will prove the converse.

As with the first inequality, the idea is to look how compacts are moved by the fundamental group. For a compact $K \subset \tilde{M}$ and $D > 0$, set

$$S_K(D) = \{\gamma \in \pi_1(M) : d(K, \gamma K) < D\}.$$

Lemma 1. *Let $K \subset \tilde{M}$ be a compact whose $\pi_1(M)$-orbit is $\tilde{M}$, and $a \in K$. Set $\delta := \operatorname{diam} M$, let $D > \delta$ and let n be such that*

$$d(a, \gamma K) < (D - \delta)n + \delta.$$

Then γ can be written as the product of n elements in $S_k(D)$

Proof : Take $y \in \gamma K$, and points $y_1, ..., y_{n+1}$ such that

$$d(a, y_1) < \delta$$
$$d(y_i, y_{i+1}) < D - \delta$$

There exists $x_i \in K, \gamma_i \in \pi_1(M)$ such that $y_i = \gamma_i(x_i)$; we can take $\gamma_1 = 1$ and $\gamma_{n+1} = \gamma$. Then

$$\gamma = (\gamma_1^{-1}\gamma_2)(\gamma_2^{-1}\gamma_3)...(\gamma_n^{-1}\gamma_{n+1}^{-1})$$

But

$$\begin{aligned} d(x_{i+1}, \gamma_i^{-1}\gamma_{i+1}(x_i)) &= d(\gamma_i(x_{i+1}), y_{i+1}) \\ &\leq d(\gamma_i(x_{i+1}), \gamma_i(x_i)) + d(\gamma_i(x_i), y_{i+1}) \\ &= d(x_{i+1}, x_i) + d(y_i, y_{i+1}) \\ &< D \end{aligned}$$

Hence $\gamma_i^{-1}\gamma_{i+1} \in S_K(D)$.
In particular, the set $S_K(D)$ is generating. To compute the growth function with respect to it, we need to show that it can be taken finite.

Lemma 2. *Let M be a compact manifold, $(\tilde{M}, p)$ its universal covering. There exists a compact $K \subset \tilde{M}$ such that (γK) is a locally finite cover of $\tilde{M}$.*

Recall that a cover is locally finite if every point has a neighbourhood intersecting only finitely many elements of the cover.

Proof : Let $(U_i, \phi_i)_{i \in I}$ be a finite cover of M by open sets which are both charts and local trivialisation for p. The collection

$$\{\phi_i^{-1}B : i \in I, B \subset \mathbb{R}^n \text{ is a closed ball }\}$$

is a collection of subset whose interiors cover M. Take (K_j) a finite subcover. For every j, choose one connected component $\tilde{K}_j$ of $p^{-1}(K_j)$, and set $\tilde{K} = \bigcup \tilde{K}_j$.

By transitivity of the monodromy action, $(\gamma\tilde{K})$ covers $\tilde{M}$. To show that it is locally finite, let $\tilde{x} \in \tilde{M}$. Let W be the intersection of every chart U_i from that contains $x = p(\tilde{x})$. Withdraw from W all the compact set K_j that intersect W without containing x, and let $\tilde{W}$ be the connected component of $p^{-1}(W)$ containing $\tilde{x}$. If $\gamma\tilde{K} \cap \tilde{W} \neq \varnothing$, then there exists j such that $\gamma\tilde{K}_j \cap \tilde{W} \neq \varnothing$. Hence $K_j \cap W \neq \varnothing$. Thus, K_j and W belong to the same trivialisation chart U_i ; hence γ maps the connected component of $p^{-1}(U_i)$ containing $\tilde{K}_i$ to that containing $\tilde{W}$. By freeness of the monodromy action, such a γ is unique.

Remark : there exists an alternative proof of this lemma, using the Riemannian structure on M. It can be found in the notes of the previous session.

Corollary 3. *With $K \subset \tilde{M}$ as in the previous lemma and $a \in K$, $S_K(D)$ is a finite generating set of $\pi_1(M)$ and*

$$\operatorname{vol} B(a, (D-\delta)n + \delta) \leq \phi_S(n) \operatorname{vol} K.$$

In particular

$$\operatorname{vol} B(a, n) \preccurlyeq \phi(n).$$

Proof : The set $S_K(D)$ is then finite, because if it were infinite, one could take a limit value of γx for $\gamma \in S_K(D)$ and some fixed $x \in K$, which would contradict local finiteness of $(\gamma K)_\gamma$.

The ball $B(a, (D-\delta)n+\delta)$ is covered by (γK), and lemma 1 shows one needs only $\phi_S(n)$ translated of K to do this.

2 Riemannian geometry

2.1 Geodesics

Last time, we defined (metric) geodesics as curves that locally minimize length. In the context of Riemannian geometry, one define geodesics rather as curves whose acceleration is zero.

To explain what acceleration of a curve is, one need a new way to derivate vector fields along curves.

Definition 4. *Let γ be a (piecewise) smooth curve $[a,b] \to M$. A **vector field** along γ is a (piecewise) smooth map $[a,b] \to TM$ such that for each $t \in [a,b]$, one has $V_t \in T_{\gamma(t)}M$. The vector space of all vector fields along γ is denoted by $\mathcal{T}(\gamma)$.*

Given a smooth curve γ, the velocity of γ is the vector field

$$t \mapsto \gamma'(t).$$

A vector field along γ which is the restriction of a globally defined vector field on M is called an **extendible vector field**. Note that in general, not all vector field along curves are extendible ; thus one cannot use the derivations already defined in the setting of smooth manifolds, such as Lie derivative. An example of non-extendible vector field is the velocity field of a non-injective curve γ with $\gamma(t_0) = \gamma(t_1)$ and $\gamma'(t_0) \neq \gamma'(t_1)$.

2.1.1 Connection

Recall that $\Gamma(TM)$ denotes the space of vector fields on M.

Définition 5. *A **connection** on a smooth manifold M is a smooth bilinear map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM); (X,Y) \mapsto \nabla_X Y$$

such that

- *∇ is $\mathcal{C}^\infty(M)$ linear in the first variable ;*
- *for f a smooth function and X, Y vector fields, it satisfies the product rule*

$$\nabla_X(fY) = f\nabla_X Y + \mathcal{L}_X(f)Y.$$

The following lemma allows one to think of $\nabla_X Y$ as the derivative of Y in the direction X.

***Proposition 6.** *The value of a connection $\nabla_X Y$ at a point p does not depend on the entire vector fields X, Y, but only on the value of X in p and the value of Y on a neighbourhood of p. In other words, if X, X' and Y, Y' are vector fields with $X_p = X'_p$ and $Y = Y'$ on an open set containing p, then $(\nabla_X Y)_p = (\nabla_{X'} Y')_p$.*

Proof : We begin by proving this for Y. By linearity, it suffices to show that if Y is a vector field that vanish on an open set $U \ni p$, then $(\nabla_X Y)(p) = 0$. Choose a bump function φ with support in U such that $\varphi(p) = 1$. Then φY is the zero vector field, so

$$\nabla_X(\varphi Y) = \nabla_X(0.\varphi X) = 0.$$

Hence, the product rule gives

$$0 = \nabla_X(\varphi Y) = \mathcal{L}_X(\varphi)Y + \varphi \nabla_X Y.$$

Evaluating at p, one find that $(\nabla_X Y)_p = 0$.

A similar technique proves that $\nabla_X Y$ depends only of the values of X in a neighbourhood of p. Let U be a chart around p, and (∂_i) the local frame. Write $X = \sum X^i \partial_i$ in U ; then

$$(\nabla_X Y)_p = \sum X^i(p)(\nabla_{\partial_i} Y)_p$$

where we have used $\mathcal{C}^\infty(M)$ linearity on the first variable. Hence $(\nabla_X Y)_p$ depends only on $X^i(p)$.

Definition 7.** *Let ∂_i be a local frame*[1]*. The* ***Christoffel symbols *Γ^k_{ij} with respect to this frame are the coefficients of $\nabla_{\partial_i} \partial_j$ in the frame :*

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{i,j} \partial_k.$$

Theorem 8. *(fundamental lemma of Riemannian geometry).*
On a Riemannian manifold (M, g), there exists a unique connection which is

- ***torsion free*** *: for all vector fields X, Y, one has*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

- ***compatible with the metric*** *: for all vector fields X, Y, Z, one has*

$$\mathcal{L}_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

It is called the ***Levi-Civita connection****.*

**Proof* : We first derive a formula for ∇ to show its uniqueness. Let X, Y, Z be vector fields whose Lie brackets are zero. Then writing the compatibility condition for cyclic permutation of X, Y, Z, one find

$$\begin{aligned}
\mathcal{L}_X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
\mathcal{L}_Y(g(Z, X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\
\mathcal{L}_Z(g(X, Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\end{aligned}$$

Using the torsion-free condition on the last term of each line and vanishing of Lie brackets, one has

$$\begin{aligned}
\mathcal{L}_X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_Z X) \\
\mathcal{L}_Y(g(Z, X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_X Y) \\
\mathcal{L}_Z(g(X, Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Y X)
\end{aligned}$$

Adding the first two of these equations and subtracting the third, one obtain

$$\mathcal{L}_X(g(Y, Z)) + \mathcal{L}_Y(g(Z, X)) - \mathcal{L}_Z(g(X, Y)) = 2g(\nabla_X Y, Z) \tag{1}$$

[1] i.e. the pull-back of the canonical vector fields on $\mathbb{R}^n$ by a chart

Now let (∂_i) be a local frame. Let $g_{ij} = g(\partial_i, \partial_j)$ be the local coefficient of the metric and Γ_{ij}^k the Christoffel symbols. As the Lie brackets of the local frame vanish, equation (1) for basis vectors $(\partial_i, \partial_j, \partial_l)$ gives :

$$\sum_k \Gamma_{i,j}^k g_{k,l} = \frac{1}{2}(\mathcal{L}_{\partial_i}(g_{j,l}) + \mathcal{L}_{\partial_j}(g_{i,l}) - \mathcal{L}_{\partial_l}(g_{i,j}))$$

Let g^{lk} be the coefficient of the inverse matrix of g ; multiplying last equation by this inverse matrix gives

$$\Gamma_{i,j}^k = \frac{1}{2}\sum_l g^{kl}(\mathcal{L}_{\partial_i}(g_{j,l}) + \mathcal{L}_{\partial_j}(g_{i,l}) - \mathcal{L}_{\partial_l}(g_{i,j}))$$

This show that a Levi-Civita connection is unique on a local frame. With lemma 6, this implies that such a connection is unique on M.

For the existence, we can define a connection on each chart U by setting, for vector fields $X = \sum X^i \partial_i, Y = \sum Y^i \partial_i$ with support in U

$$\nabla_X Y = \sum_k \left(\mathcal{L}_X(Y^k) + \sum_{i,j} X^i Y^j \Gamma_{i,j}^k \right) \partial_k$$

where the Christoffel symbols are defined by the previous formula. One can check that this give rise to a Levi-Civita connection on each chart ; by uniqueness, these constructions agree on each intersection and define a connection on M.

2.1.2 Covariant derivative

We introduce connections to make sense of directional derivative of vector fields along curves.

Proposition 9. *For each curve $\gamma : I \to M$, there exists a unique linear operator*

$$D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$$

such that

- *D_t satisfy the product rule:*
$$D_t(fV) = f'V + fD_tV$$
- *if V is extendible, then for any extension $\tilde{V}$ of V,*
$$D_tV(t) = \nabla_{\gamma'}\tilde{V}.$$

*This operator is called the **covariant derivative** along γ.*

**Proof* : ([2], 2.68) The proof is similar to that of theorem 8. Using bump functions, one obtains that the value of D_tV in t_0 depends only on the value of V on a neighbourhood of t_0. Hence, it suffices to show local uniqueness. On coordinates, write $V_t = \sum V^i(t)\partial_i(t)$. By the property of D_t, as ∂_i is extendible, one has

$$(D_tV)_{t_0} = \sum_k \left((V^k)'(t_0) + \sum_{i,j} V^j(t_0)(\gamma^i)'(t_0)\Gamma_{ij}^k(\gamma(t_0)) \right) \partial_k(t)$$

Then one can define D_tV using this formula, and check it satisfies the required properties.

Definition 10. *A (Riemannian) **geodesic** is a smooth curve γ whose acceleration is zero :*

$$D_t\gamma' \equiv 0.$$

Theorem 11. *(existence and uniqueness for geodesics)*

For any point $p \in M$, vector $V \in T_pM$ and $t_0 \in \mathbb{R}$, there exists a geodesic γ with $\gamma(t_0) = p$ and $\gamma'(t_0) = V$. Any two such geodesic agree on their common domain.

**Proof* : ([1], 4.10) Choose coordinates on some neighbourhood U of p. From the proof of theorem 9, one see that a curve $I \to U$ with component functions $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$ is a geodesic if and only if it satisfies the *geodesic equation* :

$$\gamma_k'' + \sum_{i,j} \gamma_i' \gamma_j' \Gamma_{ij}^k(\gamma) = 0.$$

This is a second-order system of linear equations, which can be rewritten as a first-order system in twice the number of variable. The theorem follows from existence and uniqueness of solution of such systems.

2.1.3 Geodesics in the model spaces

An isometry $f : M \to M$ maps a geodesic γ passing through p to the geodesic starting at $f(p)$ with initial velocity $d_p f(\gamma'(p))$. Hence one can use symmetry to determine geodesics without computations.

Sphere. The sphere $\mathbb{S}^n$ is endowed with the Riemannian metric g coming from $\mathbb{R}^{n+1}$: for any $p \in \mathbb{S}_n$ and $V, W \in T_p\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $g(V,W)$ is the usual Euclidean scalar product. Orthogonal transformations act by isometry on $\mathbb{R}^{n+1}$, so also on $\mathbb{S}^n$.

Let γ be a geodesic passing through p with velocity V. The reflection around the plane P spanned by V and p leaves γ invariant, so the image of γ have to be the great circle $P \cap \mathbb{S}^n$.

Hyperbolic space $\mathbb{H}^n$. Let $\mathbb{B}^n$ be the open unit ball. The hyperbolic metric on $\mathbb{B}^n$ is defined by

$$ds^2 = 4\frac{(dx_1)^2 + ... + (dx_n)^2}{(1 - |x|^2)^2}.$$

This means that for a point $x \in \mathbb{B}^n$ and two tangent vectors $V, W \in T_x\mathbb{B}^n$, identified with vectors in $\mathbb{R}^n$, one has

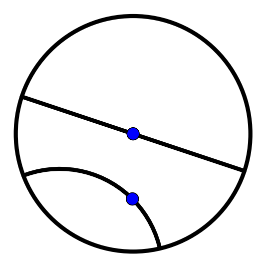
$$g_x(V,W) = 4\frac{V_1W_1 + ... + V_nW_n}{(1 - |x|^2)^2}.$$

From this expression, one see that an orthogonal transformation preserve the hyperbolic metric. Hence, by an argument similar to that of spheres, the geodesic passing through the origin are the straight Euclidean lines. Now let p, q be two points of $\mathbb{H}^n$, and P the 2-plane containing p, q and the origin. Then the reflection around this plane is an isometry, so a geodesic joining p and q must lie in P. The form of the hyperbolic metric implies that the intersection of P and $\mathbb{H}^n$, endowed with the restriction of the metric, is the 2-dimensional hyperbolic plane $\mathbb{H}^2$. Seeing $\mathbb{H}^2$ as a subset of $\mathbb{C}$, one can check that the transformations

$$f_{\theta,a} = z \mapsto e^{i\theta}\frac{z - a}{1 - \bar{a}z}$$

are isometries.

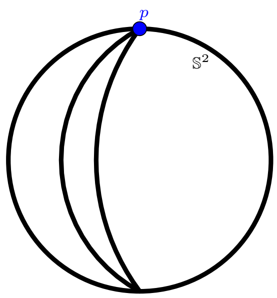
They act 2-transitively, so every geodesic can be realized as the image of a straight line by such a transformation. But straight lines are orthogonal to the boundary $\mathbb{S}^1 = \partial H^2$, and those isometries are holomorphic, so geodesics are orthogonal to $\mathbb{S}^n$. In addition, $f_{\theta,\alpha}$ is a Möbius transformation, so it maps generalized spheres (i.e. lines and circles) to generalized spheres. Hence a geodesic in $\mathbb{H}^2$ is either a straight line or the intersection of a circle that is orthogonal to $\mathbb{S}^2$.



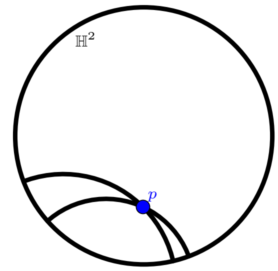
Geodesics in $\mathbb{H}^2$

2.2 Jacobi Fields

The purpose of this section is to introduce a tool that relates geodesics and curvature. Intuitively, those two notions should be linked : on a positively curved manifold, closed geodesics tend to converge, while on a negatively curved manifold, they tend to spread out.



(a) (Portions of) geodesics on the sphere



(b) Two geodesics on $\mathbb{H}^2$. If they are of equal constant speed, the (hyperbolic) distance between them at time t grows exponentially with t

First, we recall the definition of curvature.

***Definition 12.** *The Riemann curvature endomorphism is the map $R : \Gamma(TM)^2 \times \Gamma(TM) \to \Gamma(TM)$ defined by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The relation between curvature and geodesics appears when one looks at infinitesimal variations of a geodesic. To make this idea precise, we introduce the following.

Definition 13. *Let $\gamma : I \to M$ be a curve. A **variation** of γ is a smooth map*

$$\Gamma :]-\varepsilon, \varepsilon[\times I \to M$$

that agree with γ on $\{0\} \times I$.

*Given a variation Γ, its **main curves** are the curves $\Gamma_s : t \mapsto \Gamma(s, t)$ and its **transverse curves** are the curves $T_t : s \mapsto \Gamma(s, t)$. If each main curve is a geodesic, one says that Γ is a **variation through geodesics**.*

*The velocities of the transverse curves at their intersection with γ define a vector field along γ, called the **variation field** of Γ :*

$$V_t = (D_s(T_t))_{s=0}.$$

Variation fields of a variation through geodesics are characterized by a second-order differential equation.

Theorem 14. *Let Γ be a variation of γ through geodesics, V its variation field. Then*

$$D_t^2 V + R(V, \gamma')\gamma' = 0.$$

Reciprocally, every vector field along γ satisfying this equation is the variation field of a variation through geodesics.

A proof is given in chapter 10 of [1]. As the Jacobi equation amounts to a second order linear system of ODE in a chart, for every curve γ passing through p, and every vector $V, W \in T_pM$, there is a unique Jacobi field J such that $J(p) = V, (D_tJ)(p) = W$.

References

[1] John M.Lee. *Riemannian Manifolds*. Springer-Verlag, 1997.

[2] J.Lafontaine S.Gallot, D.Hulin. *Riemannian Geometry*. Springer-Verlag, 1987.