Some useful properties of Jacobi fields

We set (M, g) an n-dimensionnal Riemannian manifold with $n \geq 2$.

First recall that a Jacobi field J along a geodesic γ satisfies the Jacobi equation:

$$D_t^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0$$

definition. Let J a Jacobi field along γ, we say J is normal (to γ) if $\forall t \; J(t) \perp \dot{\gamma}(t)$. This is equivalent to $J(a) \perp \dot{\gamma}(a)$ and $D_t J(a) \perp \dot{\gamma}(a)$ for some a.

For the equivalence compute the second derivative of the scalar product using the Jacobi equation and notice it is zero.

We will need another notion of curvature called the scalar curvature:

definition. Set p a point in M and Π a plane in T_pM with basis (X, Y) and define:

$$K(X, Y) := \frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - (X, Y)^2} =: K(\Pi)$$

It does not depend on the choice of basis and is called the sectionnal curvature of M at Π.

A standard example is the hyperbolic plane with metric $\dfrac{dx^2 + dy^2}{y^2}$ which has sectionnal curvature -1 at all points. In such case we say that the manifold is of constant scetionnal curavture.

Lemma. Suppose (M, g) has constant sectionnal curvature C and γ is a unit speed geodesic. Then the normal Jacobi fields along γ vanishing at $t = 0$ are given by:

$$J(t) = u(t)E(t)$$

where E is a parallel normal vector field along γ (parallel means $D_t E = 0$) and:

$$\begin{aligned} u(t) = t \; for \; C &= 0 \\ Rsin(t/R) \; for \; C &= 1/R^2 > 0 \\ Rsinh(t/R) \; for \; C &= -1/R^2 < 0 \end{aligned}$$

Proof. This follows from the following computation of the curavture tensor when the sectional curvature is constant:

$$R(X,Y)Z = C((Y,Z)X - (X,Z)Y)$$

□

definition. Set p, q two points in M and γ a geodesic from p at $t = 0$ to q at $t = 1$. We say q is conjugated to p along γ if there exists a non zero Jacobi field J along γ such that $J(0) = J(1) = 0$. We say p and q are conjugate points if they are conjugated for some geodesic linking them.

We can now state an important lemma that partially motivates the study of Jacobi fields.

Lemma. Let $p \in M$, $V \in T_pM$ and $q = exp_pV \in M$, then $exp_p : T_pM \to M$ is a local diffeomorphism of a neighberhood of V if and only if q and p are not conjugate point.

Proof. By the inverse function theorem we need only check $(exp_p)_*$ is an isomorphism on V and it suffices to verify injectivity. Recall:

$$(exp_p)_*W|_V = \partial_s|_{s=0}exp_p(V + sW)$$

Now define $\Gamma_W(s,t) = exp_pt(V + sW)$ and the Jacobi fields $J_W(t) = \partial_s\Gamma_W(0,t)$ and notice $J_W(1) = (exp_p)_*W|_V$. □

We conclude this section with the following very useful Jacobi field comparaison theorem:

Theorem. Suppose that (M,g) has sectional curvature bounded up by a constant C, then any Jacobi field J satisfies the following inequalities:

$$\begin{aligned}|J(t)| \geq\ & t|D_tJ(0)| \ for\ C = 0\\ & Rsin(t/R)|D_tJ(0)| \ for\ C = 1/R^2 > 0\\ & Rsinh(t/R)|D_tJ(0)| \ for\ C = -1/R^2 < 0\end{aligned}$$

Proof. Compute $\frac{d^2}{dt^2}|J|$ and show using the Jacobi equation and Cauchy-Schwarz:

$$\frac{d^2}{dt^2}|J| \geq -C|J|$$

□

Cartan-Hadamard

We prove the following theorem in this section:

Theorem. (M, g) complete connected manifold with everywhere non positive sectional curvature then $exp_p : T_pM \to M$ is a covering map of Riemannian manifolds.

Proof. By the Jacobi field comparaison theorem non positive sectional curvature implies no conjugate points which implies that exp_p is a local diffeomorphism on all of T_pM. Now define $\tilde{g} := (exp_p)_* g$ then $exp_p : (T_pM, \tilde{g}) \to (M, g)$ is a local isometry. □

This suffices thanks to the following general lemma:

Lemma. $\tilde{M}$, M complete connected Riemannian manifolds and $\pi : \tilde{M} \to M$ a local isometry, then pi is a covering map (of Riemannian manifolds).

Proof. We first prove surjectivity and lifting of geodesics:

Let $\tilde{p} \in \tilde{M}$ and $p = \pi(\tilde{p})$ and $q \in M$ arbitrary. Since M is complete we can choose a geodesic γ from p to q. We now lift this geodesic as follows: note $V = \dot{\gamma}(0)$ and define $\tilde{V} = \pi_*^{-1}\dot{\gamma}(0)$ and consider the geodesic $\tilde{\gamma}$ starting at $\tilde{p}$ with velocity $\tilde{V}$. Since π is a local isometry it sends geodesics to geodesics and we have $\pi(\tilde{\gamma}(0)) = \gamma(0)$ and $\pi_*(\dot{\tilde{\gamma}}(0)) = \dot{\gamma}(0)$ so it sends $\tilde{\gamma}$ to γ and since $\tilde{M}$ is complete we can find some time t such that $\pi(\tilde{\gamma}(t)) = \gamma(t) = q$.

We now show it is indeed a covering map:

Let $p \in M$ and define $U = B_\epsilon(p)$ the geodesic ball of radius ϵ. Now note $\pi^{-1}(p) = \{\tilde{p}_\alpha\}_\alpha$ and $\tilde{U}_\alpha$ the metric ball of radius ϵ around $\tilde{p}_\alpha$. Let $\alpha \neq \beta$ and $\tilde{\gamma}$ from $\tilde{p}_\alpha$ to $\tilde{p}_\beta$ a minimising geodesic. $\pi(\tilde{\gamma})$ is a geodesic from p to p so it must leave U and reenter it so it has length at least 2ϵ and $d(\tilde{p}_\alpha, tildep_\beta) > 2\epsilon$ and $\tilde{U}_\alpha \cap \tilde{U}_\beta$ is empty.

We have $\cup_\alpha \tilde{U}_\alpha \subset \pi^{-1}(U)$ because π is a local isometry. Let $\tilde{q} \in \pi^{-1}(U)$, $q = \pi(\tilde{q})$ and γ the geodesic from q to p, it lifts to a geodesic $\tilde{\gamma}$ from $\tilde{q}$ to $\tilde{p}_\alpha$ for some α. Now compute $d(\tilde{q}, \tilde{p}_\alpha) \leq L(\tilde{\gamma}) = L(\gamma) < \epsilon$.

To conlude we need to prove that $\pi : \tilde{U}_\alpha \to U$ is a diffeomorphism for each α. We need only show bijectivity which is clear through geodesics. □