Groups and curvature

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1 Motivation : triangles in the hyperbolic plane

The purpose of this section is to derive a metric property of the hyperbolic plane that will be used to define what is a general hyperbolic space.

Proposition 1. *Every triangle in $\mathbb{H}^2$ has an area bounded by π*

This is a consequence of the Gauss-Bonnet theorem, but we can prove it directly. Recall that we use the Poincaré disk model $\mathbb{H}^2 = \mathbb{D} \subset \mathbb{C}$. The boundary $\partial\mathbb{H}^2$ is just $\mathbb{S}^1 = \partial D$. Given two points in the boundary, there is a unique geodesic whose limit is those points ; an ideal triangle is a triangle formed by geodesics connecting points of the boundary.

Lemma 2. *Hyperbolic isometries act transitively on unordained triples of distinct points of ∂H^2*

Proof : Computations are more convenient in the upper-half plane model $\{z \in \mathbb{C} : \Im(z) > 0\}$. It is isometric to $\mathbb{H}^2$ via $z \mapsto (z-i)/(z+i)$, and its isometry group is $PSL(2,\mathbb{R})$, acting by Möbius transformation. Its boundary is $\mathbb{R} \cup \{\infty\}$.

Let $\alpha, \beta, \gamma \in \partial\mathbb{H}^2$ be distinct boundary points. Without loss of generality[1], one can suppose that $\alpha, \beta, \gamma \in \mathbb{R}$. Now

$$\begin{pmatrix} 1 & -\alpha \\ \frac{\beta-\alpha}{\beta-\gamma} & -\gamma\frac{\beta-\alpha}{\beta-\gamma} \end{pmatrix}$$

sends (α, β, γ) to $(0, 1, \infty)$ and is invertible. If it has a negative determinant, exchange the roles of α and β.

Lemma 3. *The area of an ideal triangle is π.*

Proof : By the previous lemma, it suffices to compute it for a specific ideal triangle. Take the triangle whose vertices are $1, -1, i$ (in the disk model). It decomposes in two triangles along the segment $[0, i]$. Then direct integration proves the claim.

Proof (of proposition 1) : Let T be a triangle in H. By transitivity of the isometries, one can suppose that one of its vertices is 0. Then T is contained in an ideal triangle.

Area is not a metric notion, but the following is.

Corollary 4. *There exists a constant K such that for every α, β, γ vertices of a triangle in $\mathbb{H}^2$, for every p a point in the geodesic from α to β, there exists a point q at distance at most K from p on the geodesic from α to γ or from β to γ*

Proof : Consider a circle inside T that meet p and one other side of T. Then it has area less than π. By homogeneity of $\mathbb{H}^2$, there exists a constant K such that every ball having area less than π has radius less than K.

[1] use a rotation in the Poincaré disk model

2 Definition and examples

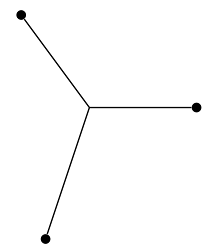
Definition 5. *Let X be a metric space. A (metric) geodesic between two points x, y is an isometry $\gamma : [0, d(x, y)] \to X$ joining x to y. The space X is* ***geodesic*** *if every two points can be joined by a geodesic.*

If x, y are two points in a geodesic space, one denotes sometimes $[x, y]$ a geodesic joining them.

Definition 6. *Let $\delta > 0$. A geodesic space X is* ***δ-hyperbolic*** *if given any three points x, y, z and any geodesics $[x, y], [y, z], [x, z]$, every point in $[x, y]$ is at distance at most δ of a point in $[y, z]$ or in $[x, z]$*

One also say that in X, triangles are δ-thin.

Examples and non-examples of hyperbolic spaces.



A triangle in a tree

1. A bounded metric space X is diam(X)-hyperbolic
2. Trees are 0-hyperbolic. Indeed, in a tree any two points are joined by a unique reduced path
3. As we have seen in the introduction, the hyperbolic plane is hyperbolic.
4. More generally, a simply connected Riemannian manifold whose sectional curvature is everywhere inferior to a fixed negative number $k < 0$ is hyperbolic ([2], chapter 3).
5. $\mathbb{R}^2$ is not hyperbolic.

Next theorem will show that hyperbolicity is a "large scale" property.

Definition 7. *Let X, Y be two metric spaces. A map $f : X \to Y$ is* ***coarse Lipschitz*** *if there exists a constant K such that for every $x, x' \in X$*

$$d(f(x), f(x')) \leq Kd(x, x') + K$$

If in addition there exists a coarse Lipschitz function $g : Y \to X$ and a constant K such that for every $x \in X$

$$d(g(f(x)), x) \leq K$$

and for every $y \in Y$

$$d(f(g(y)), y) \leq K$$

then it is a ***quasi isometry****.*

Theorem 8. *If X is a hyperbolic space quasi isometric to Y, then Y is hyperbolic.*

Note that the constant of hyperbolicity can change from X to Y. The proof of this theorem is not easy ([2], ch. 5); it could be a good topic for a future lecture.

Definition 9. *A metric space is* ***proper*** *if its compact sets are its closed bounded sets. If G acts on X, the action is* ***cocompact*** *if there is a compact set K such that $G.K = X$.*

There are lots of equivalent definitions of hyperbolic groups. We will use the following.

Proposition 10. *Let G be a group. Are equivalent :*

1. *G acts properly, cocompactly and by isometries on a proper hyperbolic space ;*
2. *G is finitely generated and one Cayley graph of G is hyperbolic ;*
3. *G is finitely generated and all its Cayley graphs are hyperbolic.*

If G satisfy one of these property, it is called ***hyperbolic****.*

Proof : 3. $\implies$ 2. $\implies$ 1. is easy. Now suppose 1. and let X be a hyperbolic space on which G acts. From the Švarc-Milnor lemma[2], G is finitely generated and any of its Cayley graph is quasi isometric to X. Since hyperbolicity is a quasi-isometry invariant, the Cayley graph is hyperbolic.

Examples and non-examples of hyperbolic groups

1. Finite groups ;
2. Free groups of finite rank : their Cayley graphs are trees ;
3. Fundamental groups of closed surface of genus $g \geq 2$. Indeed the universal covering of such a surface is the hyperbolic plane, that we have seen to be hyperbolic ; and in a previous lecture we have seen that the monodromy action is proper, cocompact and by isometries. Recall that the fundamental group of the closed surface of genus g is
$$\pi_1(\Sigma_g) =< a_1, b_1, ..., a_g, b_g | [a_1, b_1]...[a_g, b_g] = 1 >$$
4. More generally, every fundamental group of a compact Riemannian manifold having negatively-majored sectional curvature.

3 Hyperbolic groups are finitely presented

Recall that a group is said to be *finitely presented* if it is isomorphic to a quotient $\mathbb{F}_n/H$, where $\mathbb{F}_n$ is the free group with basis $a_1, ..., a_n$, and $H \triangleleft F_n$ is a subgroup that is finitely normally generated. If $h_1, ..., h_k$ are words in F_n that normally generates H, then one writes

$$G =< a_1, ..., a_n | h_1, ..., h_k > .$$

The purpose of this section is to show that hyperbolic groups are finitely presented. This will rely on the following property.

Definition 11. *Let Γ be a graph. It has the **fellow traveller property** if there exists a constant K such that given any two geodesics γ_1, γ_2 with the same endpoints, one has*

$$d(\gamma_1(t), \gamma_2(t)) \leq K$$

for all t from 0 to the length of the geodesics. A group is said to have the fellow traveller property if one of its Cayley graph has it.

Proposition 12. *Hyperbolic groups have the fellow traveller property.*

Proof : Let Γ be a δ-hyperbolic Cayley graph and $\gamma_1, \gamma_2 : [0, d] \to \Gamma$ two geodesics with the same endpoints $a := \gamma_i(0); b := \gamma_i(d)$. Let $t \in [0, d]$ and $x := \gamma_1(t), y := \gamma_2(t)$. By applying δ-hyperbolicity to the triangle formed by γ_1, γ_2^{-1} and the constant path equals to a, one gets that there exists a point z in γ_2 such that $d(x, z) \leq \delta$. Without loss of generality z lies between a and y. Then since γ_2 is a geodesic,

$$\begin{aligned} d(a, z) + d(z, y) &= d(a, y) \\ &= d(a, x) \\ &\leq d(a, z) + d(z, x) \end{aligned}$$

Hence $d(z, y) \leq \delta$ and $d(x, y) \leq 2\delta$.

Theorem 13. *Groups with the fellow traveller property are finitely presented.*

Proof : Let S be a generating set for G having the K-fellow traveller property. We will show that any word w on the letters in S is a product of conjugates of words of length at most $2K$.

We proceed by induction on the length (in the free group) of w. Since $w = 1$ in G, it defines a loop in the Cayley graph. Let x be its middle point, or one of its middle points if the length is odd. This divide w into two

[2]See the proof of theorem 1.(ii) in the first lecture notes. Note also that when we have linked the growth of the fundamental group of a compact Riemannian manifold and the growth of balls in its universal cover, we have in fact reproved the Švarc-Milnor lemma

paths, $w = w_1 w_2$. If w_1 is not a geodesic, let v be a geodesic joining its end points. Then $w = w_1 v^{-1} v w_2$ and we can use induction. If w_1, w_2 are geodesics, let x_1, x_2 be their middle points, separating $w_1 = w_3 w_4$ and $w_2 = w_5 w_6$. Then $w_3 v w_6$ and $w_4 w_5 v^{-1}$ are loops of strictly inferior length, and

$$w = (w_3 v w_6)[w_6^{-1} v^{-1} (w_4 w_5 v^{-1}) v w_6].$$

Questions :

- *Is the fellow traveller property is invariant under quasi isometry ?*
- *for a graph, the fellow traveller property is strictly weaker than hyperbolicity (take a graph consisting of arbitrary large odd circles, joined in a chain : it is uniquely geodesic, so has the fellow traveller property, but it is not hyperbolic). Does there exists a non-hyperbolic group with the fellow traveller property ?*

References

[1] E. Charpentier, E. Ghys, and A. Lesne. *The Scientific Legacy of Poincaré*. AMS, London Mathematical Society, 2010.

[2] E. Ghys and P. de la Harpe. *Hyperbolic groups*. Birkhauser, 1990.

[3] Avinoam Mann. *How Groups Grow*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2011.