Riemannian Geometry and Holonomy

Lecture Notes

Paul Lemarchand

Sorbonne Université

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1 Preliminaries on Smooth Vector Bundles

In this section, we review some basic operations on smooth vector bundles, in particular the tensor product and pullback of vector bundles, as well as tensor bundles and tensor fields.

Let E and F be two smooth vector bundles over a smooth manifold M with typical fibers V and W, respectively.

1.1 Tensor Product of Vector Bundles

The tensor product of the vector bundles E and F is a vector bundle $E \otimes F$ over M with fiber $V \otimes W$. The total space is given by

$$E \otimes F = \bigsqcup_{p \in M} (E_p \otimes F_p).$$

Given local trivializations (U, φ) for E and (U, ψ) for F, we can construct a local trivializations $(U, \varphi \otimes \psi)$ for $E \otimes F$ by defining

$$\varphi \otimes \psi : \begin{cases} \pi^{-1}(U) & \longrightarrow & U \times (V \otimes W) \\ v_p \otimes w_p & \longmapsto & (p, \phi_p(v_p) \otimes \psi_p(w_p)). \end{cases}$$

Lemma 1.1. *There is a canonical isomorphism of $\mathbb{C}^\infty(M)$-modules*

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F).$$

In particular, the sections $\Gamma(E \otimes F)$ of the tensor product bundle $E \otimes F$ is generated, as a $C^\infty(M)$-module, by sections of the form $s \otimes t$ with $s \in \Gamma(E)$ and $t \in \Gamma(F)$.

Proof. The desired map

$$\Phi : \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) \longrightarrow \Gamma(E \otimes F),$$

is defined on simple tensors by

$$s \otimes t \longmapsto (p \mapsto s_p \otimes t_p).$$

We leave it to the reader to verify that Φ is a well-defined $C^\infty(M)$-linear isomorphism. □

1.2 Pullback Bundle

Let $f : N \to M$ be a smooth map between smooth manifolds. The pullback bundle f^*E is a vector bundle over N with fiber V. The total space is given by

$$f^*E = \bigsqcup_{q \in N} E_{f(q)}.$$

Given a local trivialization (U, φ) for E, we can construct a local trivialization $(f^{-1}(U), f^*\varphi)$ for f^*E by defining

$$f^*\varphi : \begin{cases} \pi^{-1}(f^{-1}(U)) & \longrightarrow & f^{-1}(U) \times V \\ (q, v_{f(q)}) & \longmapsto & (q, \phi_{f(q)}(v_{f(q)})). \end{cases}$$

Lemma 1.2. *Let $f : N \to M$ be a smooth map between smooth manifolds, and let E be a smooth vector bundle over M. There is a canonical isomorphism of $C^\infty(N)$-modules*

$$\Gamma(f^*E) \cong C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E).$$

Remark . Note that the tensor product $C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E)$ is taken over the ring $C^\infty(M)$, where the $C^\infty(M)$-module structure on $C^\infty(N)$ is induced by the pullback map $f^* : C^\infty(M) \to C^\infty(N), h \mapsto h \circ f$. Moreover, this tensor product is a naturally equipped with a $C^\infty(N)$-module structure via multiplication on the first factor, i.e., on simple tensors,

$$g \cdot (h \otimes s) = (gh) \otimes s, \qquad g, h \in C^\infty(N), s \in \Gamma(E).$$

Proof. The desired map

$$\Phi : C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E) \longrightarrow \Gamma(f^*E),$$

is defined on simple tensors by

$$g \otimes s \longmapsto g \cdot (s \circ f)$$

We leave it to the reader to verify that Φ is a well-defined $C^\infty(N)$-linear isomorphism.

Another way of seeing this isomorphism is to use the sheaf-theoretic description of sections. It suffices to construct a sheaf isomorphism

$$\Phi : C_N^\infty \otimes_{f^{-1}C_M^\infty} f^{-1}\mathcal{E} \xrightarrow{\sim} \mathcal{F},$$

and the desired isomorphism on global sections follows by taking $\Gamma(N, -)$. Here, C_N^∞ and C_M^∞ denote the sheaves of smooth functions on N and M respectively, $\mathcal{E}$ denotes the sheaf of sections of E, and $\mathcal{F}$ denotes the sheaf of sections of f^*E. □

1.3 Tensor Bundle and Tensor Fields

We now introduce tensor bundles and tensor fields, which are fundamental objects in Riemannian geometry.

Definition 1.3. The *tensor bundle* of type (r, s) over a smooth manifold M is the smooth vector bundle $\mathcal{T}_s^r M$ defined by

$$\mathcal{T}_s^r M := (TM)^{\otimes r} \otimes (T^*M)^{\otimes s} = \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ times}}.$$

Its smooth sections are called tensor fields of type (r, s). The integers r and s are called the *covariant degree* and *contravariant degree* respectively.

More generally, if E is a smooth vector bundle over M, we define the *tensor bundle* of type (r, s) with values in E by $\mathcal{T}_s^r M \otimes E$. Its smooth sections are called tensor fields of type (r, s) with values in E.

Remark . Note that if $E = M \times \mathbb{R}$ is the trivial line bundle over M, then $\Gamma(E) \cong C^\infty(M)$. Hence, tensor fields of type (r, s) with values in the trivial line bundle are simply tensor fields of type (r, s).

Moreover, if $E = TM$ is the tangent bundle, then tensor fields of type (r, s) with values in TM are tensor fields of type $(r, s + 1)$. Similarly, if $E = T^*M$ is the cotangent bundle, then tensor fields of type (r, s) with values in T^*M are tensor fields of type $(r + 1, s)$.

Lemma 1.4. *Let $T \in \Gamma(\mathcal{T}^r_s M \otimes E)$ be a tensor field of type (r, s) with values in E. Let $U \subseteq M$ be a coordinate neighborhood with local coordinates $(x^1, \dots, x^n)$, and let $(e_1, \dots, e_k)$ be a local frame of $\Gamma(E|_U)$. On U, the section T can be uniquely written as*

$$T|_U = T^{i_1\dots i_r,\alpha}_{j_1\dots j_s}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes e_\alpha, \quad T^{i_1\dots i_r,\alpha}_{j_1\dots j_s} \in C^\infty(U),$$

where indices i_k, j_ℓ run from 1 to n, and α runs from 1 to k (with the Einstein summation convention understood).
Moreover, if $V \subseteq M$ is another coordinate neighborhood with local coordinates $(\tilde{x}^1, \dots, \tilde{x}^n)$, and if $(\tilde{e}_1, \dots, \tilde{e}_k)$ is a local frame of $\Gamma(E|_V)$, then on the overlap $U \cap V$, the components of T transform according to the rule

$$T^{i_1\dots i_r,\alpha}_{j_1\dots j_s} = \frac{\partial x^{i_1}}{\partial \tilde{x}^{k_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{k_r}} \frac{\partial \tilde{x}^{l_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{l_s}}{\partial x^{j_s}} g^\alpha_\beta \tilde{T}^{k_1\dots k_r,\beta}_{l_1\dots l_s},$$

where $g^\alpha_\beta \in C^\infty(U \cap V)$ are the components of the transition function $g : U \cap V \to \mathrm{GL}(n, \mathbb{R})$ of the vector bundle E with respect to the local frames $(e_1, \dots, e_k)$ and $(\tilde{e}_1, \dots, \tilde{e}_k)$.

Proof. The existence and uniqueness of the local expression of T follows from the fact that $\{\partial_i\}$ and $\{dx^j\}$ form local frames of $TM|_U$ and $T^*M|_U$ respectively, and $\{e_\alpha\}$ is a local frame of $\Gamma(E|_U)$. The components $T^{i_1\dots i_r,\alpha}_{j_1\dots j_s}$ are smooth functions on U because T is a smooth section.

To derive the transformation law, we express the local frames in V in terms of those in U. The change of coordinates gives

$$\tilde{\partial}_k = \frac{\partial x^i}{\partial \tilde{x}^k} \partial_i, \quad d\tilde{x}^l = \frac{\partial \tilde{x}^l}{\partial x^j}\, dx^j,$$

and the change of frame for the vector bundle E gives

$$\tilde{e}_\beta = g^\alpha_\beta e_\alpha.$$

Substituting these into the expression for $T|_V$, we have

$$\begin{aligned} T|_{U\cap V} &= \tilde{T}^{k_1\dots k_r,\alpha}_{l_1\dots l_s}\, \tilde{\partial}_{k_1} \otimes \cdots \otimes \tilde{\partial}_{k_r} \otimes d\tilde{x}^{l_1} \otimes \cdots \otimes d\tilde{x}^{l_s} \otimes \tilde{e}_\beta \\ &= \tilde{T}^{k_1\dots k_r,\alpha}_{l_1\dots l_s} \left(\frac{\partial x^{i_1}}{\partial \tilde{x}^{k_1}} \partial_{i_1} \right) \otimes \cdots \otimes \left(\frac{\partial x^{i_r}}{\partial \tilde{x}^{k_r}} \partial_{i_r} \right) \otimes \left(\frac{\partial \tilde{x}^{l_1}}{\partial x^{j_1}}\, dx^{j_1} \right) \otimes \cdots \otimes \left(\frac{\partial \tilde{x}^{l_s}}{\partial x^{j_s}}\, dx^{j_s} \right) \otimes (g^\alpha_\beta e_\alpha) \\ &= \frac{\partial x^{i_1}}{\partial \tilde{x}^{k_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{k_r}} \frac{\partial \tilde{x}^{l_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{l_s}}{\partial x^{j_s}} g^\alpha_\beta \tilde{T}^{k_1\dots k_r,\beta}_{l_1\dots l_s}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes e_\alpha. \end{aligned}$$

By uniqueness of the local expressions, we can compare this with the expression for $T|_U$, and we obtain the desired transformation law for the components. □

Theorem 1.5. *Let M be a smooth manifold and (E, M, π, V) be a vector bundle over M. Then, any tensor fields of type (r, s) with values in E identifies with $C^\infty(M)$-multilinear maps*

$$\underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{r \text{ times}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s \text{ times}} \to \Gamma(E).$$

More precisely, there is a canonical isomorphism of $C^\infty(M)$-modules

$$\Gamma(\mathcal{T}^r_s M \otimes E) \cong \mathrm{Mult}_{C^\infty(M)}(\Omega^1(M)^r \times \mathfrak{X}(M)^s, \Gamma(E)).$$

Proof. Let us denote the desired map by

$$\Psi : \Gamma(\mathcal{T}^r_s M \otimes E) \longrightarrow \mathrm{Mult}_{C^\infty(M)}(\Omega^1(M)^r \times \mathfrak{X}(M)^s, \Gamma(E)).$$

By definition of the tensor bundle, a section $T \in \Gamma(\mathcal{T}^r_s M)$ assigns to each point $p \in M$ an element $T(p) \in (\mathcal{T}^r_s)_p M \otimes E_p$. Using the standard correspondence between tensor products and multilinear maps (see lemma **??**), we can identify

$$(\mathcal{T}^r_s)_p M \otimes E_p \cong \mathrm{Mult}_{\mathbb{R}}(T_pM^r \times T^*_pM^s, E_p).$$

Hence, it is natural to define

$$\Psi(T) : \begin{cases} \Omega^1(M)^r \times \mathfrak{X}(M)^s & \longrightarrow \quad \Gamma(E) \\ (\omega^1, \dots, \omega^r, X_1, \dots, X_s) & \longmapsto \quad (p \mapsto T(p)(\omega^1(p), \dots, \omega^r(p), X_1(p), \dots, X_s(p))) \end{cases}$$

To check the smoothness, let $U \subseteq M$ be a coordinate neighborhood with local coordinates $(x^1, \dots, x^n)$, and let $(e_1, \dots, e_k)$ be a local frame of $\Gamma(E|_U)$. On U, the section $T|_U$ can be uniquely written as

$$T|_U = T^{i_1\dots i_r,\alpha}_{j_1\dots j_s}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes e_\alpha, \quad T^{i_1\dots i_r,\alpha}_{j_1\dots j_s} \in C^\infty(U).$$

Now, take r one-forms $\omega^1, \dots, \omega^r \in \Omega^1(M)$ and s vector fields $X_1, \dots, X_s \in \mathfrak{X}(M)$, and write them in the local frame

$$\omega^k = \omega^k_i\, dx^i, \quad X_l = X^j_l \partial_j, \quad \omega^k_i, X^j_l \in C^\infty(U).$$

Then, on U, we have

$$\Psi(T)(\omega^1, \dots, \omega^r, X_1, \dots, X_s)|_U = T^{i_1\dots i_r,\alpha}_{j_1\dots j_s} \omega^1_{i_1} \cdots \omega^r_{i_r} X^{j_1}_1 \cdots X^{j_s}_s e_\alpha,$$

which is a smooth function on U as a sum of product of smooth functions times the local frame. Since M can be covered by such coordinate neighborhoods, $\psi(T)(\omega^1, \dots, X_s) \in \Gamma(E)$ is globally smooth. Moreover, $\Psi(T)$ is $C^\infty(M)$-multilinear because $T(p)$ is $\mathbb{R}$-multilinear for each $p \in M$. Similarly, one can check that Ψ is $C^\infty(M)$-linear. Thus we obtain a natural $C^\infty(M)$-linear map.

Hence, it remains to show that Ψ is an isomorphism. To see that Ψ is injective, suppose that $\Psi(T) = 0$. Then, for all $p \in M$ and all $\xi^i \in T^*_pM$, $v_j \in T_pM$, we have

$$T(p)(\xi^1, \dots, \xi^r, v_1, \dots, v_s) = 0 \in E_p.$$

Since $T(p)$ is a multilinear map, this implies that $T(p) = 0$ for all $p \in M$, and hence $T = 0$.

Now, to see that Ψ is surjective, let

$$L \in \mathrm{Mult}_{C^\infty(M)}(\Omega^1(M)^r \times \mathfrak{X}(M)^s, \Gamma(E)).$$

We construct a section $T \in \Gamma(\mathcal{T}^r_s M)$ locally. Choose a coordinate neighborhood $U \subseteq M$ with local coordinates $(x^1, \dots, x^n)$, and choose a local frame $(e_1, \dots, e_k)$ of $\Gamma(E|_U)$. Then, on U, we can write (uniquely)

$$L(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) = T^{i_1\dots i_r,\alpha}_{j_1\dots j_s} e_\alpha, \quad T^{i_1\dots i_r,\alpha}_{j_1\dots j_s} \in C^\infty(U),$$

so that the components $T^{i_1\dots i_r,\alpha}_{j_1\dots j_s}$ define a smooth section of $\mathcal{T}^r_s M|_U \otimes E|_U$.

To show that these local sections glue to give a global section $T \in \Gamma(\mathcal{T}^r_s M \otimes E)$, it suffices to check that the components transform according to the transformation law of tensor fields. For this, we use another coordinate neighborhood $V \subseteq M$ with local coordinates $(\tilde{x}^1, \dots, \tilde{x}^n)$, and let $(\tilde{e}_1, \dots, \tilde{e}_k)$ be a local frame of $\Gamma(E|_V)$. On V, we can similarly write

$$L(d\tilde{x}^{k_1}, \dots, d\tilde{x}^{k_r}, \tilde{\partial}_{l_1}, \dots, \tilde{\partial}_{l_s}) = \tilde{T}^{k_1\dots k_r,\beta}_{l_1\dots l_s} \tilde{e}_\beta, \quad \tilde{T}^{k_1\dots k_r,\beta}_{l_1\dots l_s} \in C^\infty(V).$$

On the overlap $U \cap V$, we can write the change of coordinates as

$$\tilde{\partial}_l = \frac{\partial x^j}{\partial \tilde{x}^l} \partial_j, \quad d\tilde{x}^k = \frac{\partial \tilde{x}^k}{\partial x^j}\, dx^j,$$

and the change of frame for the vector bundle E gives

$$\tilde{e}_\beta = g^\alpha_\beta e_\alpha,$$

where $g^\alpha_\beta \in C^\infty(U \cap V)$ are the components of the transition function $g : U \cap V \to \mathrm{GL}(n, \mathbb{R})$ of the vector bundle E with respect to the local frames $(e_1, \dots, e_k)$ and $(\tilde{e}_1, \dots, \tilde{e}_k)$.

Then, we can express $L(d\tilde{x}^{k_1}, \dots, d\tilde{x}^{k_r}, \tilde{\partial}_{l_1}, \dots, \tilde{\partial}_{l_s})$ in terms of the local coordinates on U as

$$\begin{aligned}
L(d\tilde{x}^{k_1}, \dots, d\tilde{x}^{k_r}, \tilde{\partial}_{l_1}, \dots, \tilde{\partial}_{l_s}) &= L\left(\frac{\partial \tilde{x}^{k_1}}{\partial x^{i_1}} dx^{i_1}, \dots, \frac{\partial \tilde{x}^{k_r}}{\partial x^{i_r}} dx^{i_r}, \frac{\partial x^{j_1}}{\partial \tilde{x}^{l_1}} \partial_{j_1}, \dots, \frac{\partial x^{j_s}}{\partial \tilde{x}^{l_s}} \partial_{j_s}\right) \\
&= \frac{\partial \tilde{x}^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial \tilde{x}^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \tilde{x}^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial \tilde{x}^{l_s}} L(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) \\
&= \frac{\partial \tilde{x}^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial \tilde{x}^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \tilde{x}^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial \tilde{x}^{l_s}} T^{i_1 \dots i_r, \alpha}_{j_1 \dots j_s} e_\alpha.
\end{aligned}$$

On the other hand, when expressed in terms of the local frame on $\Gamma(E|_U)$, we have

$$L(d\tilde{x}^{k_1}, \dots, d\tilde{x}^{k_r}, \tilde{\partial}_{l_1}, \dots, \tilde{\partial}_{l_s}) = \tilde{T}^{k_1 \dots k_r, \beta}_{l_1 \dots l_s} g^\alpha_\beta e_\alpha.$$

Comparing these two expressions, we obtain the desired transformation law for the components. Thus, the local sections glue to give a global section $T \in \Gamma(\mathcal{T}^r_s M \otimes E)$ such that $\Psi(T) = L$. This shows that Ψ is surjective. □

2 Connections on Vector Bundles

A connection on a vector bundle generalizes the notion of directional derivatives (or differentiation of vector fields) from Euclidean space to curved spaces or more general manifolds.

Definition 2.1. A *connection*, or *covariant derivative*, on a smooth vector bundle $E \to M$ is a $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Equivalently, by theorem 1.5, a connection can be viewed as a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s := (\nabla s)(X).$$

satisfying the following properties for all $X, Y \in \mathfrak{X}(M)$, $s, t \in \Gamma(E)$, and $f \in C^\infty(M)$:

(i) $\nabla_{fX+Y} s = f\nabla_X s + \nabla_Y s$ ($C^\infty(M)$-linearity in the vector field argument),

(ii) $\nabla_X(s + t) = \nabla_X s + \nabla_X t$ (linearity in the section argument),

(iii) $\nabla_X(fs) = X(f)s + f\nabla_X s$ (Leibniz rule in the section argument).

The Leibniz rule implies that ∇ is a local operator, i.e., the value of ∇s at a point $p \in M$ depends only on the germ of s at p. More precisely, we have the following proposition.

Proposition 2.2. *Let ∇ be a connection on a smooth vector bundle $E \to M$. If $s_1, s_2 \in \Gamma(E)$ are two sections such that $s_1|_U = s_2|_U$ on some open set $U \subseteq M$, then*

$$(\nabla s_1)|_U = (\nabla s_2)|_U.$$

Proof. By linearity of ∇, it suffices to show that if s vanishes on U, then ∇s vanishes on U. Fix $p \in U$, and choose a bump function $f \in C^\infty(M)$ such that $f \equiv 1$ on some neighborhood of p and $\operatorname{supp}(f) \subseteq U$. Since s vanishes on U and f vanishes outside U, the product fs must vanish on M. Applying the Leibniz rule,

$$0 = \nabla(fs) = df \otimes s + f\nabla s.$$

Evaluating at p, we get

$$0 = (df)_p \otimes s(p) + f(p)(\nabla s)(p) = 1 \cdot (\nabla s)(p),$$

which shows that $(\nabla s)(p) = 0$. Since p was arbitrary, we conclude that ∇s vanishes on U. □

Lemma 2.3. *Let ∇ be a connection on a smooth vector bundle $E \to M$. If $X_1, X_2 \in \mathfrak{X}(M)$ are two vector fields such that $X_1|_p = X_2|_p$ at some point $p \in M$, then for any section $s \in \Gamma(E)$,*

$$(\nabla_{X_1} s)_p = (\nabla_{X_2} s)_p$$

Proof. By linearity of ∇ in the vector field argument, it suffices to show that if X vanishes at p, then $\nabla_X s$ vanishes at p. By definition, $\nabla s \in \Gamma(T^*M \otimes E)$, so $(\nabla s)_p : T_pM \to E_p$ is linear. Thus, if $X_p = 0$, then

$$(\nabla_X s)_p = (\nabla s)_p(X_p) = (\nabla s)_p(0) = 0,$$

as desired. □

This lemma shows that the value of $\nabla_X s$ at a point p only depends on the value of the vector field X at p. In particular, this means that it makes sense to write $\nabla_{X_p} s$ for any tangent vector $X_p \in T_pM$: one simply chooses a vector field X with $X|_p = X_p$ and defines

$$\nabla_{X_p} s \coloneqq (\nabla_X s)_p.$$

2.1 Local Expression of a Connection

Let E be a smooth vector bundle over a smooth manifold M, and let ∇ be a connection on E. Let $U \subseteq M$ be a coordinate neighborhood with local coordinates $(x^1, \dots, x^n)$, and let $(e_1, \dots, e_k)$ be a local frame of $\Gamma(E|_U)$.

Definition 2.4. The *Christoffel symbols* of ∇ with respect to the local frame $(e_1, \dots, e_k)$ are the smooth functions $\Gamma_{ij}^k \in C^\infty(U)$ defined by

$$\nabla e_j = \Gamma_{ij}^k \, dx^j \otimes e_k,$$

or equivalently,

$$\nabla_{\partial_i} e_j = \Gamma_{ij}^k e_k.$$

More compactly, we can define $\Gamma_i \in \Gamma(\operatorname{End}(E|_U))$ by $\Gamma_i(e_j) = \Gamma_{ij}^k e_k$, so that

$$\nabla e_j = dx^i \otimes \Gamma_i(e_j).$$

2.2 Pullback of Connections

When pulling back a vector bundle via a smooth map, one can also pull back a connection on the original bundle to obtain a connection on the pullback bundle in a natural way.

Proposition 2.5. *Let ∇ be a connection on a smooth vector bundle $E \to M$, and let $f : N \to M$ be a smooth map. Then, there is a unique connection $f^*\nabla$ on the pullback bundle $f^*E \to N$ such that*

$$(f^*\nabla)_X(s \circ f) = f^*(\nabla_{df(X)}s),$$

for all $X \in \mathfrak{X}(N)$ and $s \in \Gamma(E)$.
Moreover, for any smooth map $g : P \to N$, the pullback connection satisfies

$$(f \circ g)^*\nabla = g^*(f^*\nabla).$$

Remark . In the statement of the lemma, the notation $f^*(\nabla_{df(X)}s)$ must be interpreted carefully, because $df(X)$ is not a vector field in general. What is really meant is that for each point $p \in N$,

$$(f^*(\nabla_{df(X)}s))_p := (\nabla_{d_pf(X_p)}s)_{f(p)} \in E_{f(p)} = (f^*E)_p,$$

where we use Lemma 2.3 to ensure it is well-defined. Note that if f were a diffeomorphism, then $df(X)$ would define a vector field on M, and the notation $f^*(\nabla_{df(X)}s)$ would indeed correspond to the above formula. However, in general, it should always be understood pointwise at each $p \in N$.

Proof. Using the canonical isomorphism of $C^\infty(N)$-modules (see lemma 1.2)

$$\Phi : C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E) \xrightarrow{\sim} \Gamma(f^*E), \qquad \Phi(h \otimes s) = h(s \circ f),$$

we can conveniently define the pullback connection.

For each $X \in \mathfrak{X}(N)$, let

$$D_X : C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E) \longrightarrow C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E)$$

be the $\mathbb{R}$-linear map given on simple tensors by

$$D_X(h \otimes s) = X(h) \otimes s + h \otimes \nabla_{df(X)}s.$$

A direct computation using the Leibniz rule for ∇ shows that D_X respects the balancing relations: for all $g \in C^\infty(M)$, one has

$$\begin{aligned} D_X(h \otimes (gs)) &= X(h) \otimes (gs) + h \otimes \nabla_{df(X)}(gs) \\ &= (X(h)(g \circ f)) \otimes s + (hX(g \circ f)) \otimes s + (h(g \circ f)) \otimes \nabla_{df(X)}s \\ &= X(h(g \circ f)) \otimes s + (h(g \circ f)) \otimes \nabla_{df(X)}s \\ &= D_X((h(g \circ f)) \otimes s), \end{aligned}$$

hence D_X is well defined on the tensor product.

We then define a $\mathbb{R}$-linear map $(f^*\nabla)_X : \Gamma(f^*E) \to \Gamma(f^*E)$ by

$$(f^*\nabla)_X := \Phi \circ D_X \circ \Phi^{-1}.$$

On a generator $h(s \circ f) \in \Gamma(f^*E)$ this gives

$$(f^*\nabla)_X(h(s \circ f)) = X(h)(s \circ f) + hf^*(\nabla_{df(X)}s),$$

and in particular $(f^*\nabla)_X(s \circ f) = f^*(\nabla_{df(X)}s)$, as desired.

To verify the Leibniz rule, it suffices to check it on sections of the form $h(s \circ f)$ by linearity in the section argument. For $g \in C^\infty(N)$, we have

$$\begin{aligned} (f^*\nabla)_X(gh(s \circ f)) &= (f^*\nabla)_X((gh)(s \circ f)) \\ &= X(gh)(s \circ f) + ghf^*(\nabla_{df(X)}s) \\ &= X(g)h(s \circ f) + g(X(h)(s \circ f) + hf^*(\nabla_{df(X)}s)) \\ &= X(g)h(s \circ f) + g(f^*\nabla)_X(h(s \circ f)). \end{aligned}$$

Thus $f^*\nabla$ defines a connection on f^*E.

Finally, the connection is uniquely determined by the given formula: if $\tilde{\nabla}$ is another connection on f^*E such that $\tilde{\nabla}_X(s \circ f) = f^*(\nabla_{df(X)}s)$ for all $s \in \Gamma(E)$, then by the Leibniz rule and the fact that sections of the form $s \circ f$ generate $\Gamma(f^*E)$ as a $C^\infty(N)$-module, we must have $\tilde{\nabla} = \nabla$.

For the second part, let $g : P \to N$ be a smooth map. For any $X \in \mathfrak{X}(P)$ and $s \in \Gamma(E)$, we have

$$\begin{aligned}(g^*(f^*\nabla))_X(s \circ f \circ g) &= g^*((f^*\nabla)_{dg(X)}(s \circ f)) \\ &= g^*(f^*(\nabla_{df(dg(X))}s)) \\ &= (f \circ g)^*(\nabla_{d(f\circ g)(X)}s) \\ &= ((f \circ g)^*\nabla)_X(s \circ f \circ g),\end{aligned}$$

which shows that $(f \circ g)^*\nabla = g^*(f^*\nabla)$ by the uniqueness of the pullback connection. □

3 Affine Connections

Let M be a smooth manifold with a connection ∇ on the tangent bundle TM. Such a connection is called an *affine connection* on M. Since $\Gamma(TM) = \mathfrak{X}(M)$, we can view ∇ as a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y.$$

3.1 Extension to Tensor Bundles

It is reasonable to ask whether an affine connection on M extends to a connection on all tensor bundles over M in a natural and compatible way. This is indeed the case, as stated in the following lemma.

Lemma 3.1. *Given an affine connection ∇ on M, there is exists a unique connection on each tensor bundle $\mathcal{T}^r_s M$, also denoted by ∇, satisfying the following property:*

(i) Compatibility with the natural pairing: For all $X, Y \in \mathfrak{X}(M)$, and $\omega \in \Omega^1(M)$,

$$X\langle\omega, Y\rangle = \langle\nabla_X\omega, Y\rangle + \langle\omega, \nabla_X Y\rangle.$$

(ii) Leibniz rule for tensor products: For all $X \in \mathfrak{X}(M)$, $T \in \Gamma(\mathcal{T}^r_s M)$, and $S \in \Gamma(\mathcal{T}^{r'}_{s'} M)$,

$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S).$$

Proof. To show existence, we first define the connection on 1-forms. The property (i) forces us to define, for all $X, Y \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$,

$$(\nabla_X\omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y).$$

One must check that this definition indeed gives a 1-form: for $f \in C^\infty(M)$,

$$\begin{aligned}(\nabla_X\omega)(fY) &= X(\omega(fY)) - \omega(\nabla_X(fY)) \\ &= X(f)\omega(Y) + fX(\omega(Y)) - X(f)\omega(Y) - f\omega(\nabla_X Y) \\ &= f(\nabla_X\omega)(Y),\end{aligned}$$

which shows that $\nabla_X\omega : \mathfrak{X}(M) \to C^\infty(M)$ is $C^\infty(M)$-linear, hence a 1-form. Moreover, one can verify that ∇ satisfies the properties of a connection: it is clearly $C^\infty(M)$-linear in the vector field argument and linear in the 1-form argument, so it remains to check the Leibniz rule in the 1-form argument. For $f \in C^\infty(M)$, we have

$$\begin{aligned}(\nabla_X(f\omega))(Y) &= X(f\omega)(Y) - f\omega(\nabla_X Y) \\ &= X(f)\omega(Y) + fX(\omega(Y)) - f\omega(\nabla_X Y) \\ &= X(f)\omega(Y) + f(\nabla_X\omega)(Y),\end{aligned}$$

as desired.

Next, we define the connection on general tensor fields by using property (ii). For simple tensors $T = \omega^1 \otimes \cdots \otimes \omega^r \otimes Y_1 \otimes \cdots \otimes Y_s$, with $\omega^i \in \Omega^1(M)$ and $Y_j \in \mathfrak{X}(M)$, we set

$$\begin{aligned}\nabla_X T = &\sum_{i=1}^{r} \omega^1 \otimes \cdots \otimes \nabla_X \omega^i \otimes \cdots \otimes \omega^r \otimes Y_1 \otimes \cdots \otimes Y_s \\ &+ \sum_{j=1}^{s} \omega^1 \otimes \cdots \otimes \omega^r \otimes Y_1 \otimes \cdots \otimes \nabla_X Y_j \otimes \cdots \otimes Y_s.\end{aligned}$$

One can check that this definition extends uniquely to all tensor fields by linearity, and that the resulting map $\nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{T}^r_s M) \to \Gamma(\mathcal{T}^r_s M)$ satisfies the properties of a connection.

Note with this construction, the connection on a general (r, s)-tensor field $T \in \Gamma(\mathcal{T}^r_s M)$ can be expressed as

$$\begin{aligned}(\nabla_X T)(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s) = &X(T(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s)) \\ &+ \sum_{i=1}^{r} T(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^r, Y_1, \ldots, Y_s) \\ &- \sum_{j=1}^{s} T(\omega^1, \ldots, \omega^r, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_s),\end{aligned}$$

Uniqueness follows from properties (i) and (ii) together with the fact that tensor fields can be expressed as linear combinations of tensor products of vector fields and 1-forms. □

3.2 Torsion and Curvature

With an affine connection ∇ on M, we can define two important tensor fields: the torsion tensor and the curvature tensor.

Definition 3.2. The *torsion tensor* of ∇ is the $(1, 2)$-tensor field T defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \mathfrak{X}(M)$. We say that ∇ is *torsion-free* if $T \equiv 0$.

Lemma 3.3. *The torsion tensor T of ∇ is indeed a $(1, 2)$-tensor field. Moreover, T is skew-symmetric, i.e., $T(X, Y) = -T(Y, X)$.*

Proof. It suffices to show that T is a $(0, 2)$-tensor field with values in TM, or equivalently, that $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is $C^\infty(M)$-linear in both arguments. Indeed, for the first argument, we have

$$\begin{aligned}T(fX, Y) &= \nabla_{fX} Y - \nabla_Y(fX) - [fX, Y] \\ &= f\nabla_X Y - Y(f)X - f\nabla_Y X + Y(f)X \\ &= f(\nabla_X Y - \nabla_Y X - [X, Y]) = fT(X, Y).\end{aligned}$$

For the second argument, linearity follows directly from the skew-symmetry of T. □

Definition 3.4. The *curvature tensor* of ∇ is the $(1,3)$-tensor field R defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for all $X,Y,Z \in \mathfrak{X}(M)$. We say that ∇ is *flat* if $R \equiv 0$.

Lemma 3.5. *The curvature tensor R of ∇ is indeed a $(1,3)$-tensor field. Moreover, R is skew-symmetric in the first two arguments, i.e., $R(X,Y)Z = -R(Y,X)Z$.*

Proof. It suffices to show that R is a $(0,3)$-tensor field with values in TM, or equivalently, that $R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ is $C^\infty(M)$-linear in all arguments. Indeed, for the first argument, we have

$$\begin{aligned}
R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z \\
&= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X}Z \\
&= f\nabla_X\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z + Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z \\
&= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) = fR(X,Y)Z.
\end{aligned}$$

For the second argument, linearity follows directly from the skew-symmetry of R in the first two arguments. For the third argument, we have

$$\begin{aligned}
R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\
&= \nabla_X(Y(f)Z + f\nabla_Y Z) - \nabla_Y(X(f)Z + f\nabla_X Z) - [X,Y](f)Z - f\nabla_{[X,Y]}Z \\
&= X(Y(f))Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X\nabla_Y Z \\
&\quad - Y(X(f))Z - X(f)\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z \\
&\quad - [X,Y](f)Z - f\nabla_{[X,Y]}Z \\
&= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\
&= fR(X,Y)Z,
\end{aligned}$$

as desired □

3.3 Levi-Civita Connection

Given a Riemannian manifold (M,g), it is natural to ask whether there exists an affine connection on M that is compatible with the metric g in some sense. The following definition makes this notion precise.

Definition 3.6. An affine connection ∇ on a Riemannian manifold (M,g) is said to be *metric compatible* if $\nabla g = 0$, i.e., for all $X,Y,Z \in \mathfrak{X}(M)$,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

More generally, an affine connection ∇ on a smooth manifold M is said to be *compatible* with a tensor field T if $\nabla T = 0$.

Lemma 3.7. *Let (M,g) be an oriented Riemannian manifold with Riemannian volume form vol_g. If ∇ is compatible with g, then ∇ is compatible with vol_g.*

Proof. Let $(e_1,\ldots,e_n)$ be a local orthonormal frame of TM over an open set $U \subseteq M$ compatible with the orientation, so that $\mathrm{vol}_g(e_1, \dots, e_n) = 1$. For any $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}(\nabla_X \mathrm{vol}_g)(e_1, \dots, e_n) &= X(\mathrm{vol}_g(e_1, \dots, e_n)) - \sum_{i=1}^{n} \mathrm{vol}_g(e_1, \dots, \nabla_X e_i, \dots, e_n) \\ &= -\sum_{i=1}^{n} \mathrm{vol}_g(e_1, \dots, \nabla_X e_i, \dots, e_n).\end{aligned}$$

Since ∇ is metric compatible, we have for each i,

$$0 = X(g(e_i, e_i)) = 2g(\nabla_X e_i, e_i),$$

which implies that $\nabla_X e_i$ is orthogonal to e_i. Therefore,

$$\mathrm{vol}_g(e_1, \dots, \nabla_X e_i, \dots, e_n) = 0,$$

because an alternating form vanishes when two arguments are linearly dependent. Hence,

$$(\nabla_X \mathrm{vol}_g)(e_1, \dots, e_n) = 0.$$

Since this holds for any local orthonormal frame and vol_g is determined by its values on such frames, we conclude that $\nabla_X \mathrm{vol}_g = 0$ for all $X \in \mathfrak{X}(M)$. □

Theorem 3.8 (Levi-Civita). *Given a Riemannian manifold (M, g), there exists a unique affine connection on M that is both torsion-free and metric compatible. This connection is called the* Levi-Civita connection *of (M, g).*

Proof. Let ∇ be a connection on TM. The condition for ∇ to be metric gives three equations

$$\begin{aligned}X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ Y(g(X, Z)) &= g(\nabla_Y X, Z) + g(X, \nabla_Y Z), \\ Z(g(X, Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y).\end{aligned}$$

Adding the first two equations and subtracting the third, we get

$$\begin{aligned}&X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\qquad = g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X).\end{aligned}$$

The condition for ∇ to be torsion-free gives three more equations

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \nabla_X Z - \nabla_Z X = [X, Z], \quad \nabla_Y Z - \nabla_Z Y = [Y, Z].$$

Substituting this into the above equation, we obtain

$$\begin{aligned}&X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\qquad = 2g(\nabla_X Y, Z) - g([X, Y], Z) + g([X, Z], Y) + g([Y, Z], X).\end{aligned}$$

Rearranging gives the *Koszul formula*

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)\Big).$$

Since g is non-degenerate, this formula uniquely determines $\nabla_X Y$ for all $X, Y \in \mathfrak{X}(M)$. Indeed, the non-degeneracy of g implies that for each fixed X, Y, the map $Z \mapsto g(\nabla_X Y, Z)$ is a linear functional on T_pM for each $p \in M$, and thus there exists a unique vector $\nabla_X Y(p) \in T_pM$ such that this holds for all Z. Varying p, we obtain a unique vector field $\nabla_X Y$. This shows the uniqueness of the Levi-Civita connection. However, we need to show the smoothness.

It is straightforward to verify that the connection defined by this formula is indeed torsion-free and metric compatible. □

4 Covariant Derivative along a Curve

Let M be a smooth manifold, and let $E \to M$ be a smooth vector bundle over M equipped with a connection ∇. We would like to define the notion of *covariant derivative* of sections of E along a smooth curve $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval.

Concretely, a section of E along γ is a smooth map

$$s : I \to E, \quad s(t) \in E_{\gamma(t)} \text{ for all } t \in I.$$

Equivalently, one can view s as a smooth section of the pullback bundle $\gamma^*E \to I$,

$$s \in \Gamma(\gamma^*E),$$

as its fibers are precisely $(\gamma^*E)_t = E_{\gamma(t)}$ for $t \in I$.

With this perspective in place, we can now define the covariant derivative of such sections along γ in a way that is compatible with the connection ∇ on E.

Definition 4.1. The *covariant derivative along* γ with respect to ∇ is the unique $\mathbb{R}$-linear operator

$$\frac{D}{dt} : \Gamma(\gamma^*E) \to \Gamma(\gamma^*E), \quad s \mapsto \frac{Ds}{dt}$$

characterized by the following two properties:

(i) Leibniz Rule: For all $f \in C^\infty(I)$ and $s \in \Gamma(\gamma^*E)$,

$$\frac{D(fs)}{dt} = \frac{df}{dt}s + f\frac{Ds}{dt}.$$

(ii) Compatibility with ∇: If $s(t) = \sigma_{\gamma(t)}$ for some $\sigma \in \Gamma(E)$, then

$$\frac{Ds}{dt}(t) = (\nabla_{\dot\gamma(t)}\sigma)_{\gamma(t)}, \quad \text{for all } t \in I.$$

The covariant derivative along γ is also denoted by $\nabla_{\dot\gamma}$.

Remark . In the case where ∇ is an affine connection on M, i.e., $E = TM$, the covariant derivative along γ is defined on vector fields along γ. This definition can be extended to tensor fields along γ since the connection ∇ extends naturally to tensor bundles over M.

Before proving that the covariant derivative along a curve exists and is unique, let us recall how the pullback connection is characterized. By proposition 2.5, there is a unique connection $\gamma^*\nabla$ on the pullback bundle γ^*E that satisfies

$$(\gamma^*\nabla)_X(s \circ \gamma) = \gamma^*(\nabla_{d\gamma(X)}s)$$

for all $X \in \mathfrak{X}(I)$ and $s \in \Gamma(E)$. We will use this fact in the proof below.

Theorem 4.2. *The definition of the covariant derivative along a curve is well-defined, i.e., there exists a unique operator $\frac{D}{dt}$ satisfying properties (i) and (ii).*

Proof. On the 1-dimensional manifold I, there is a canonical nowhere-vanishing vector field ∂_t. Define the covariant derivative along γ by

$$\frac{D}{dt} : \Gamma(\gamma^*E) \longrightarrow \Gamma(\gamma^*E), \qquad \frac{Ds}{dt} := (\gamma^*\nabla)_{\partial_t}s,$$

where $\gamma^*\nabla$ is the pullback connection.

Since $\gamma^*\nabla$ is a connection, this operator is $\mathbb{R}$-linear and satisfies the Leibniz rule. Concretely, for any $f \in C^\infty(I)$ and $s \in \Gamma(\gamma^*E)$, we have

$$\frac{D(fs)}{dt} = (\gamma^*\nabla)_{\partial_t}(fs) = \frac{df}{dt}s + f(\gamma^*\nabla)_{\partial_t}s = \frac{df}{dt}s + f\frac{Ds}{dt}.$$

so property (i) holds.

For property (ii), take $\sigma \in \Gamma(E)$ and consider the section along γ given by $s(t) = \sigma_{\gamma(t)} = \sigma \circ \gamma$. Using the defining property of the pullback connection with $X = \partial_t$ and $d\gamma(\partial_t) = \dot{\gamma}(t)$, we get

$$\frac{Ds}{dt} = (\gamma^*\nabla)_{\partial_t}(\sigma \circ \gamma) = \gamma^*(\nabla_{\dot{\gamma}}\sigma),$$

and evaluating at t yields

$$\frac{Ds}{dt}(t) = (\nabla_{\dot{\gamma}(t)}\sigma)_{\gamma(t)},$$

which is property (ii).

For uniqueness, let $\frac{D}{dt}$ and $\frac{\tilde{D}}{dt}$ be two $\mathbb{R}$-linear operators satisfying properties (i) and (ii). By property (ii), both operators agree with $(\gamma^*\nabla)_{\partial_t}$ on sections of the form $\sigma \circ \gamma$. By property (i) and $\mathbb{R}$-linearity, they then extend uniquely to all sections of γ^*E, because such sections are generated as a $C^\infty(I)$-module by sections of the form $\sigma \circ \gamma$.

Since $(\gamma^*\nabla)_{\partial_t}$ is uniquely determined, any $\mathbb{R}$-linear operator satisfying (i) and (ii) must coincide with it. This proves uniqueness. □

4.1 Properties of the Covariant Derivative along a Curve

Like the usual derivative, the covariant derivative along a curve satisfies a chain rule with respect to reparametrizations of the curve, or more generally, smooth maps between intervals.

Proposition 4.3. *Let $\gamma : I \to M$ be a smooth curve, and let $\phi : J \to I$ be a smooth map between intervals. For any section $s \in \Gamma(\gamma^*E)$, we have*

$$\frac{D(s \circ \phi)}{dt} = \left(\frac{Ds}{dt} \circ \phi\right)\frac{d\phi}{dt}.$$

Remark . Note that the covariant derivative on the left-hand side is taken along the curve $\gamma \circ \phi : J \to M$, and the one on the right-hand side is taken along the curve $\gamma : I \to M$.

Proof. By the definition of the covariant derivative along a curve using the pullback connection, we have

$$\begin{aligned}
\frac{D(s \circ \phi)}{dt}(t) &= ((\gamma \circ \phi)^*\nabla)_{\partial_t}(s \circ \phi) \\
&= (\phi^*(\gamma^*\nabla))_{\partial_t}(s \circ \phi) \\
&= \phi^*((\gamma^*\nabla)_{d\phi(\partial_t)}s) \\
&= \phi^*((\gamma^*\nabla)_{\dot{\phi}(t)\partial_t}s) \\
&= \dot{\phi}(t)\phi^*((\gamma^*\nabla)_{\partial_t}s) \\
&= \dot{\phi}(t)\left(\frac{Ds}{dt} \circ \phi\right)(t),
\end{aligned}$$

which establishes the desired chain rule. □

It also satisfies a product rule when applied to tensor fields along the curve.

Proposition 4.4. *Let ∇ be an affine connection on M, and let T be a $(0, s)$-tensor field on M. For any smooth curve $\gamma : I \to M$ and vector fields $X_1, \ldots, X_s$ along γ, we have*

$$\frac{d}{dt} T(X_1, \ldots, X_s) = (\nabla_{\dot\gamma} T)(X_1, \ldots, X_s) + \sum_{i=1}^{s} T(X_1, \ldots, \nabla_{\dot\gamma} X_i, \ldots, X_s).$$

Remark . Here we use the notation $\nabla_{\dot\gamma}$ to denote the covariant derivative along the curve γ.

Proof. Rewrite the equation to be proved in terms of the rigorous pullback notation:

$$\partial_t(T(X_1, \ldots, X_s)) = ((\gamma^*\nabla)_{\partial_t} T)(X_1, \ldots, X_s) + \sum_{i=1}^{s} T(X_1, \ldots, (\gamma^*\nabla)_{\partial_t} X_i, \ldots, X_s).$$

But this is exactly the definition of how the pullback connection $\gamma^*\nabla$ acts on tensor fields along γ. Therefore, the proposition follows directly. □

4.2 Local Expression for Covariant Derivative along a Curve

Before stating the local formula, we introduce the notation. Let $(U, x^1, \ldots, x^n)$ be a local chart on M, and write

$$\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t)), \qquad \gamma^i := x^i \circ \gamma \in C^\infty(\gamma^{-1}(U)).$$

Choose a local frame $\{e_1, \ldots, e_r\}$ of E over U. Any section s along γ can then be written uniquely as

$$s(t) = \sum_{j=1}^{r} s^j(t) e_j|_{\gamma(t)}, \qquad s^j \in C^\infty(\gamma^{-1}(U)).$$

The connection is given in this frame by

$$\nabla_{\partial_i} e_j = \sum_{k=1}^{r} \Gamma_{ij}^k e_k, \quad \Gamma_{ij}^k \in C^\infty(U).$$

With this notation, we can now state the local expression for the covariant derivative along a curve.

Proposition 4.5. *The covariant derivative of s along γ is given in local coordinates by*

$$\frac{Ds}{dt}(t) = \sum_{k=1}^{r} \left(\frac{ds^k}{dt}(t) + \sum_{i=1}^{n} \sum_{j=1}^{r} \Gamma_{ij}^k|_{\gamma(t)} \dot\gamma^i(t) s^j(t) \right) e_k|_{\gamma(t)},$$

for all $t \in \gamma^{-1}(U)$.

Remark . In matrix form, if $s(t)$ is represented by the column vector $s(t) = (s^1(t), \ldots, s^r(t))^T$ and $A(t)$ is the matrix with entries $A_j^k(t) = \sum_{i=1}^n \Gamma_{ij}^k|_{\gamma(t)} \dot\gamma^i(t)$, then the covariant derivative can be expressed as

$$\frac{Ds}{dt} = \frac{ds}{dt} + As.$$

Under a change of local frame, the matrix A transforms according to the gauge transformation law for connections, i.e.,

$$A' = P^{-1} A P + P^{-1} \frac{dP}{dt},$$

where $P(t)$ is the transition matrix between the two frames along γ, i.e., $e'_j|_{\gamma(t)} = P(t) e_j|_{\gamma(t)}$.

Proof. Using the definition of the covariant derivative along γ and the local expression for the connection, we have

$$\begin{aligned}\frac{Ds}{dt}(t) &= (\gamma^*\nabla)_{\partial_t} s(t)\\ &= (\gamma^*\nabla)_{\partial_t}\left(\sum_{j=1}^r s^j(t) e_j|_{\gamma(t)}\right)\\ &= \sum_{j=1}^r \frac{ds^j}{dt}(t) e_j|_{\gamma(t)} + \sum_{j=1}^r s^j(t)(\gamma^*\nabla)_{\partial_t} e_j|_{\gamma(t)}.\end{aligned}$$

Next, we compute $(\gamma^*\nabla)_{\partial_t} e_j|_{\gamma(t)}$. By the definition of the pullback connection,

$$(\gamma^*\nabla)_{\partial_t} e_j|_{\gamma(t)} = (\nabla_{\dot\gamma(t)} e_j)|_{\gamma(t)} = \sum_{i=1}^n \dot\gamma^i(t)(\nabla_{\partial_i} e_j)|_{\gamma(t)} = \sum_{k=1}^r\left(\sum_{i=1}^n \Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\right) e_k|_{\gamma(t)},$$

where we used the local expression for $\nabla_{\partial_i} e_j$. Combining these, we find

$$\frac{Ds}{dt}(t) = \sum_{k=1}^r\left(\frac{ds^k}{dt}(t) + \sum_{j=1}^r\sum_{i=1}^n \Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)s^j(t)\right) e_k|_{\gamma(t)}.$$

This completes the proof. □

5 Parallel Transport

With the notion of covariant derivative along a curve established, we can now introduce the notion of parallel transport along curves.

Definition 5.1. Given a smooth curve $\gamma : I \to M$, a section $s \in \Gamma(\gamma^*E)$ is called *parallel* along γ if its covariant derivative vanishes identically, i.e., $\frac{Ds}{dt} = 0$.

Lemma 5.2. *The property of being parallel along a curve is intrinsic, i.e., it does not depend on the choice of parametrization of the curve.*

Proof. Let $\gamma : I \to M$ be a smooth curve, and let $\phi : J \to I$ be reparametrization. Consider the reparametrized curve $\tilde\gamma = \gamma \circ \phi : J \to M$.

Let $s \in \Gamma(\gamma^*E)$ be a section along γ, and consider the corresponding section along $\tilde\gamma$ given by $\tilde s = s \circ \phi \in \Gamma(\tilde\gamma^*E)$. Using the chain rule for covariant derivatives along curves, we have

$$\frac{D\tilde s}{dt} = \left(\frac{Ds}{dt}\circ\phi\right)\frac{d\phi}{dt}.$$

Thus, $\frac{D\tilde s}{dt} = 0$ if and only if $\frac{Ds}{dt} = 0$. Therefore, the property of being parallel is independent of the parametrization of the curve. □

Theorem 5.3. *Given $e_0 \in E_{\gamma(t_0)}$ for some $t_0 \in I$, there exists a unique parallel section $s \in \Gamma(\gamma^*E)$ along γ such that $s(t_0) = e_0$.*

Proof. For uniqueness, let $s, s' \in \Gamma(\gamma^*E)$ be parallel along γ with $s(t_0) = s'(t_0)$, and set $u := s - s'$. Then $\nabla_{\dot\gamma} u = 0$ and $u(t_0) = 0$. Choose a Riemannian metric $\langle\cdot,\cdot\rangle$ on γ^*E, and consider the smooth function

$$f(t) := \langle u(t), u(t)\rangle_{\gamma(t)}.$$

Its derivative is

$$f'(t) = 2\langle \nabla_{\dot\gamma} u(t), u(t)\rangle_{\gamma(t)} = 0,$$

so f is constant on I. Since $f(t_0) = \langle u(t_0), u(t_0)\rangle_{\gamma(t_0)} = 0$, we get $f \equiv 0$, hence $u \equiv 0$. Thus $s = s'$.

Now, to show existence, it suffices to prove it locally, since being parallel is an intrindic property and we just proved that this property determines the section uniquely.

In local coordinates $(x^1, \dots, x^n)$ and a local frame $\{e_1, \dots, e_r\}$ of E, a section along γ can be written as

$$s(t) = \sum_{j=1}^{r} s^j(t)\, e_j|_{\gamma(t)}.$$

The parallel transport equation $\nabla_{\dot\gamma} s = 0$ then takes the form of a linear system of ODEs

$$\dot s(t) + A(t)s(t) = 0, \qquad s(t_0) = s_0,$$

where $A(t)$ is the smooth $r \times r$ matrix with entries $A^k_j(t) = \sum_{i=1}^n \Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)$, and s_0 is the coordinate vector of e_0 in the chosen frame.

By the standard existence and uniqueness theorem for such differential system, there exists a unique solution $s(t)$ defined on some interval containing t_0. By covering the interval I with such local solutions and using uniqueness to glue them together, we obtain a global parallel section along γ satisfying the initial condition. This completes the proof. □

This theorem allows us to define the parallel transport map along a curve.

Definition 5.4. Given a smooth curve $\gamma : [a, b] \to M$, the *parallel transport map* from $\gamma(a)$ to $\gamma(b)$ along γ is the map

$$P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}, \quad e \mapsto s(b),$$

where $s \in \Gamma(\gamma^* E)$ is the unique parallel section along γ with initial condition $s(a) = e$.

Note that the parallel transport map can be defined for piecewise smooth curves by applying the above definition on each smooth segment and composing the resulting maps, i.e., if $\gamma : [a, b] \to M$ is piecewise smooth with breakpoints $a = t_0 < t_1 < \cdots < t_k = b$, then

$$P_\gamma = P_{\gamma|_{[t_{k-1}, t_k]}} \circ \cdots \circ P_{\gamma|_{[t_0, t_1]}}.$$

Remark . The parallel transport map P_γ has the following properties:

(i) It is independent of the choice of parametrization of the curve γ; thus, we will suppose that it is parametrized by $[0, 1]$ from now on.

(ii) It is a linear isomorphism between the fibers $E_{\gamma(a)}$ and $E_{\gamma(b)}$.

(iii) If γ is a constant curve at a point $p \in M$, then P_γ is the identity map on E_p.

(iv) If $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$ are two piecewise smooth curves such that $\gamma_1(1) = \gamma_2(0)$, then the parallel transport along the concatenated curve $\gamma_2 * \gamma_1$ satisfies

$$P_{\gamma_2 * \gamma_1} = P_{\gamma_2} \circ P_{\gamma_1}.$$

6 Holonomy of a Connection

With the notion of parallel transport in place, we can now define the holonomy group associated with a connection on a vector bundle.

Definition 6.1. The *holonomy group* of ∇ at a point $p \in M$, denoted $\mathrm{Hol}_p(\nabla)$, is the subgroup of $\mathrm{GL}(E_p)$ defined by

$$\mathrm{Hol}_p(\nabla) = \{P_\gamma : E_p \to E_p \mid \gamma : [0,1] \to M \text{ is a piecewise smooth loop based at } p\}.$$

Lemma 6.2. *The holonomy group* $\mathrm{Hol}_p(\nabla)$ *is indeed a subgroup of* $\mathrm{GL}(E_p)$.

Proof. The fact that $\mathrm{Hol}_p(\nabla)$ is a subgroup follows from the properties of parallel transport established earlier. Specifically, the identity element of $\mathrm{Hol}_p(\nabla)$ corresponds to the constant loop at p, which yields the identity map on E_p. The composition property of parallel transport along concatenated loops ensures closure under composition, and the existence of inverses is guaranteed by considering the reverse loop. □

Lemma 6.3. *For any two path-connected points $p, q \in M$, the holonomy groups* $\mathrm{Hol}_p(\nabla)$ *and* $\mathrm{Hol}_q(\nabla)$ *are conjugate subgroups of* $\mathrm{GL}(E_p)$ *and* $\mathrm{GL}(E_q)$*, respectively. More precisely, if $\gamma : [0,1] \to M$ is a piecewise smooth curve from p to q, then*

$$\mathrm{Hol}_q(\nabla) = P_\gamma \; \mathrm{Hol}_p(\nabla) \, P_\gamma^{-1}.$$

Remark . Normally, two subgroups are called conjugate when they lie in the same group. When the subgroups lie in different groups, $H \leq \mathrm{GL}(V)$ and $K \leq \mathrm{GL}(W)$, we say that they are conjugate if there exists a linear isomorphism $f : V \to W$ such that

$$K = f \circ H \circ f^{-1}.$$

Here, we are using the parallel transport map $P_\gamma : E_p \to E_q$ as the isomorphism.

Proof. It suffices to show the inclusion $\mathrm{Hol}_q(\nabla) \subseteq P_\gamma \, \mathrm{Hol}_p(\nabla) P_\gamma^{-1}$, as the reverse inclusion follows by considering the reverse curve $\bar{\gamma}$ from q to p (since $P_{\bar{\gamma}} = P_\gamma^{-1}$) and applying the same argument.

Thus, we need to show that for any $h \in \mathrm{Hol}_p(\nabla)$, the element $P_\gamma \circ h \circ P_\gamma^{-1}$ lies in $\mathrm{Hol}_q(\nabla)$. Since h is an element of the holonomy group at p, it can be represented as P_{γ_0} for some piecewise smooth loop $\gamma_0 : [0,1] \to M$ based at p. By considering the concatenated loops $\bar{\gamma} * \gamma_0 * \gamma$, we obtain a piecewise smooth loop based at q. The parallel transport along this loop is an element of $\mathrm{Hol}_q(\nabla)$, and is given by

$$P_{\bar{\gamma} * \gamma_0 * \gamma} = P_\gamma \circ P_{\gamma_0} \circ P_{\bar{\gamma}} = P_\gamma \circ h \circ P_\gamma^{-1}.$$

This shows the desired inclusion. □

7 Riemannian Holonomy

Given a Riemannian metric g on M, there is a natural connection on the tangent bundle TM, called Levi-Civita connection, which is torsion-free and metric-compatible, and it is the unique one satisfying this. Using it, we define the holonomy of a Riemannian manifold.

Definition 7.1. The *Riemannian holonomy group* of g at a point $p \in M$ is defined as

$$\mathrm{Hol}_p(g) = \mathrm{Hol}_p(\nabla),$$

where ∇ is the Levi-Civita connection on the tangent bundle TM.

Proposition 7.2. *For each $p \in M$, the Riemannian holonomy group satisfies*

$$\mathrm{Hol}_p(g) \subseteq O(T_pM, g_p),$$

that is, every parallel transport map P_γ along a loop γ based at p is an isometry of (T_pM, g_p). If moreover M is orientable, then

$$\mathrm{Hol}_p(g) \subseteq SO(T_pM, g_p),$$

so the parallel transports P_γ preserve orientation.

Proof. Let $\gamma : [0,1] \to M$ be a piecewise smooth loop with $\gamma(0) = \gamma(1) = p$. Given $v, w \in T_pM$, extend them to vector fields V, W along γ which are parallel, i.e. $\nabla_{\dot\gamma}V = \nabla_{\dot\gamma}W = 0$, and satisfy $V(0) = v$, $W(0) = w$. Using metric-compatibility of the Levi-Civita connection, we compute the derivative (see proposition 4.4)

$$\frac{d}{dt}g(V,W) = (\nabla_{\dot\gamma}g)(V,W) + g(\nabla_{\dot\gamma}V, W) + g(V, \nabla_{\dot\gamma}W) = 0.$$

Hence, $t \mapsto g(V(t), W(t))$ is constant, so

$$g_p(v,w) = g_{\gamma(1)}(V(1), W(1)) = g_p(P_\gamma v, P_\gamma w).$$

This shows that P_γ preserves the inner product g_p, so $P_\gamma \in O(T_pM, g_p)$.

Now, if M is orientable, we can choose an orientation on M and consider the associated Riemannian volume form $\omega \in \Omega^n(M)$. Since the Levi-Civita connection is compatible with the metric, it also preserves the volume form, i.e., $\nabla\omega = 0$ (see lemma 3.7). Given a positively oriented orthonormal basis $e_1, \ldots, e_n$ of T_pM, we have $\omega_p(e_1, \ldots, e_n) = 1$. Extend these vectors to parallel vector fields $E_1, \ldots, E_n$ along γ with $E_i(0) = e_i$. Using a similar argument as above, we compute

$$\frac{d}{dt}\omega(E_1, \ldots, E_n) = (\nabla_{\dot\gamma}\omega)(E_1, \ldots, E_n) + \sum_{i=1}^{n} \omega(E_1, \ldots, \nabla_{\dot\gamma}E_i, \ldots, E_n) = 0,$$

so that $t \mapsto \omega(E_1(t), \ldots, E_n(t))$ is constant. Evaluating at $t = 0$ and $t = 1$, we find

$$\omega_p(e_1, \ldots, e_n) = \omega_p(P_\gamma e_1, \ldots, P_\gamma e_n).$$

This implies that $\det(P_\gamma) = 1$, so $P_\gamma \in \mathrm{SO}(T_pM, g_p)$. This completes the proof. □