Riemannian Geometry and Holonomy

Presentation for Working Group on Hyperkähler Geometry

Paul Lemarchand

Sorbonne Université

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Preliminaries on Smooth Vector Bundles

Recall basic operations on smooth vector bundles:

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- Pullback of bundles
- Tensor bundles and tensor fields

Tensor Product of Bundles

Let $E \to M$ and $F \to M$ be smooth vector bundles.

Definition

The tensor product bundle $E \otimes F$ is defined by:

$$E \otimes F = \bigsqcup_{p \in M} (E_p \otimes F_p)$$

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There is a $C^\infty(M)$-module isomorphism of sections:

$$\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) \cong \Gamma(E \otimes F)$$

given by

$$s \otimes t \mapsto (p \mapsto s_p \otimes t_p)$$

Pullback Bundle

Let $f : N \to M$ be a smooth map and $E \to M$ a smooth vector bundle.

Definition

The pullback bundle $f^*E \to N$ is defined by:

$$f^*E = \bigsqcup_{q \in N} E_{f(q)}$$

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given by

$$g \otimes s \mapsto g \cdot (s \circ f)$$

Tensor Bundles and Tensor Fields

Let M be a smooth manifold.

Definition

Tensor bundle of type (r, s):

$$\mathcal{T}^r_s M = (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$$

Sections are called tensor fields of type (r, s)

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- More generally: $\mathcal{T}^r_s M \otimes E$ for a vector bundle E
- Sections: $\Gamma(\mathcal{T}^r_s M \otimes E)$ are E-valued tensor fields of type (r, s)

Local Expressions and Transformation Law

- Locally: $T \in \Gamma(\mathcal{T}^r_s M \otimes E)$:

$$T|_U = T^{i_1 \ldots i_r, \alpha}_{j_1 \ldots j_s} \, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes e_\alpha$$

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  where (x^i) are local coordinates and (e_α) a local frame of E over U
- Transformation under coordinates/frames:

$$T^{i_1 \dots i_r, \alpha}_{j_1 \dots j_s} = \frac{\partial x^{i_1}}{\partial \tilde{x}^{k_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{k_r}} \frac{\partial \tilde{x}^{l_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{l_s}}{\partial x^{j_s}} g^\alpha_\beta \tilde{T}^{k_1 \dots k_r, \beta}_{l_1 \dots l_s},$$

  where g^α_β are the transition functions of E

Tensor Fields as Multilinear Maps

Theorem (Multilinear Correspondence)

$$\Gamma(\mathcal{T}^r_s M \otimes E) \cong \mathrm{Mult}_{C^\infty(M)}\big(\Omega^1(M)^r \times \mathfrak{X}(M)^s, \Gamma(E)\big)$$

Tensor Fields as Multilinear Maps

Theorem (Multilinear Correspondence)

$$\Gamma(\mathcal{T}^r_s M \otimes E) \cong \mathrm{Mult}_{C^\infty(M)}(\Omega^1(M)^r \times \mathfrak{X}(M)^s, \Gamma(E))$$

Concretely, for $T \in \Gamma(\mathcal{T}^r_s M \otimes E)$:

$$T : \begin{cases} \Omega^1(M)^r \times \mathfrak{X}(M)^s & \to & \Gamma(E) \\ (\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) & \mapsto & (p \mapsto T_p(\omega^1_p, \ldots, \omega^r_p, X_1|_p, \ldots, X_s|_p)) \end{cases}$$

Connections on Smooth Vector Bundles

Definition of a Connection

Definition

A *connection* (or *covariant derivative*) on $E \to M$ is a $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule: for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$\nabla(fs) = df \otimes s + f\nabla s$$

Equivalent Definition of a Connection

By the multilinear correspondence, a connection can be viewed as a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s \coloneqq (\nabla s)(X).$$

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satisfying:for all $X, Y \in \mathfrak{X}(M)$, $s, t \in \Gamma(E)$, and $f \in C^\infty(M)$:

(i) $\nabla_{fX+Y} s = f \nabla_X s + \nabla_Y s$

(ii) $\nabla_X (s + t) = \nabla_X s + \nabla_X t$

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1. (i) $\nabla_{fX+Y} s = f\nabla_X s + \nabla_Y s$
2. (ii) $\nabla_X(s + t) = \nabla_X s + \nabla_X t$
3. (iii) $\nabla_X(fs) = X(f)s + f\nabla_X s$

Locality of Connections

Lemma

If $s_1|_U = s_2|_U$ *on an open set* $U \subseteq M$*, then*

$$(\nabla s_1)|_U = (\nabla s_2)|_U.$$

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Idea of proof:

- Use bump functions to localize sections
- Leibniz rule ensures vanishing outside support implies vanishing of ∇s

Tensoriality of Connections

Lemma

For $X_1, X_2 \in \mathfrak{X}(M)$ *with* $X_1|_p = X_2|_p$, *and* $s \in \Gamma(E)$*:*

$$(\nabla_{X_1} s)_p = (\nabla_{X_2} s)_p$$

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Proof: Suffices to show that if $X|_p = 0$, then $(\nabla_X s)_p = 0$.

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- $\nabla s \in \Gamma(T^*M \otimes E)$, so $(\nabla s)_p : T_pM \to E_p$ is linear

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- Thus if $X|_p = 0$,

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- Thus if $X|_p = 0$,

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Consequences:

- Value of $\nabla_X s$ at p depends only on $X_p \in T_pM$
- We then write $\nabla_{X_p} s := (\nabla_X s)_p$

Local Expression of a Connection (Christoffel Symbols)

Let $(e_1, \ldots, e_k)$ be a local frame on $U \subseteq M$ with coordinates $(x^1, \ldots, x^n)$.

Definition

Christoffel symbols $\Gamma_{ij}^k \in C^\infty(U)$ are defined by

$$\nabla e_j = \Gamma_{ij}^k \; dx^i \otimes e_k,$$

or equivalently,

$$\nabla_{\partial_i} e_j = \Gamma_{ij}^k e_k.$$

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Let $f : N \to M$ be smooth, and ∇ a connection on $E \to M$.

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Theorem

There exists a unique connection $f^*\nabla$ on $f^*E \to N$ such that

$$(f^*\nabla)_X(s \circ f) = f^*(\nabla_{df(X)}s), \quad X \in \mathfrak{X}(N), s \in \Gamma(E).$$

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Moreover, for any $g : P \to N$,

$$(f \circ g)^*\nabla = g^*(f^*\nabla).$$

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Notation:

$$(f^*(\nabla_{df(X)}s))_p := (\nabla_{d_pf(X_p)}s)_{f(p)} \in E_{f(p)} = (f^*E)_p$$

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- Use the isomorphism $\Gamma(f^*E) \cong C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E)$
- For each $X \in \mathfrak{X}(N)$, define a $\mathbb{R}$-linear map

$$D_X : C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E) \longrightarrow C^\infty(N) \otimes_{C^\infty(M)} \Gamma(E)$$

  given on simple tensors by

$$D_X(h \otimes s) = X(h) \otimes s + h \otimes \nabla_{df(X)} s.$$

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$$(f^*\nabla)_X \coloneqq \Phi \circ D_X \circ \Phi^{-1}.$$

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- One checks that $(f^*\nabla)_X$ satisfies the Leibniz rule, hence defines a connection on f^*E.

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- One checks that $(f^*\nabla)_X$ satisfies the Leibniz rule, hence defines a connection on f^*E.
- Uniqueness follows from the defining property.

Affine Connections on Manifolds

Let M be a smooth manifold.

Definition

An *affine connection* on M is a connection on the tangent bundle TM:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y.$$

Extension to Tensor Bundles

Lemma

Any affine connection ∇ *on* TM *extends uniquely to a connection on all tensor bundles* $\mathcal{T}^r_s M$ *such that:*

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1. *Compatibility with pairing: for $\omega \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$,*

$$X\langle \omega, Y\rangle = \langle \nabla_X \omega, Y\rangle + \langle \omega, \nabla_X Y\rangle$$

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2. *Leibniz rule: for all $T \in \Gamma(\mathcal{T}^{r_1}_{s_1} M)$ and $S \in \Gamma(\mathcal{T}^{r_2}_{s_2} M)$,*

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Idea of construction:

- Define ∇ on 1-forms using compatibility with vector fields.

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Idea of construction:

- Define ∇ on 1-forms using compatibility with vector fields.
- Extend to general tensors by using decomposition into simple tensors and applying Leibniz rule.

Explicit Formula for the Extension

For $T \in \Gamma(\mathcal{T}^r_s M)$, the covariant derivative is given by

$$\begin{aligned}
&(\nabla_X T)(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s) \\
&\quad = X\big(T(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s)\big) \\
&\qquad + \sum_{i=1}^{r} T(\omega^1, \dots, \nabla_X \omega^i, \dots, \omega^r, Y_1, \dots, Y_s) \\
&\qquad - \sum_{j=1}^{s} T(\omega^1, \dots, \omega^r, Y_1, \dots, \nabla_X Y_j, \dots, Y_s),
\end{aligned}$$

for all $\omega^1, \dots, \omega^r \in \Omega^1(M)$ and $Y_1, \dots, Y_s \in \mathfrak{X}(M)$.

Torsion of a Connection

Definition

The *torsion tensor* T of ∇ is

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

∇ is *torsion-free* if $T \equiv 0$.

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Lemma

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To see this, prove that T is a $(0,2)$-tensor with values in TM, and use the multilinear correspondence.

Curvature of a Connection

Definition

The *curvature tensor* R of ∇ is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

∇ is *flat* if $R \equiv 0$.

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Lemma

R is a $(1,3)$-tensor and skew-symmetric in the first two arguments.

To see this, prove that R is a $(1,3)$-tensor with values in TM, and use the multilinear correspondence.

Metric Compatibility

Definition

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An affine connection ∇ on a Riemannian manifold (M, g) is *metric compatible* if $\nabla g = 0$, i.e.,

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More generally, ∇ is compatible with a tensor T if $\nabla T = 0$.

Lemma

Metric compatibility implies compatibility with the Riemannian volume form vol_g*.*

Levi-Civita Connection

Theorem (Levi-Civita)

For any Riemannian manifold (M, g)*, there exists a unique affine connection* ∇ *that is*

- *Torsion-free:* $T \equiv 0$*,*
- *Metric compatible:* $\nabla g = 0$*.*

This is called the Levi-Civita connection.

Levi-Civita Connection

Theorem (Levi-Civita)

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- *Torsion-free:* $T \equiv 0$,
- *Metric compatible:* $\nabla g = 0$.

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Koszul formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\Big).$$

Covariant Derivative along a Curve

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Equivalently, $s \in \Gamma(\gamma^* E)$, the pullback bundle along γ.

Covariant Derivative along a Curve

Definition

The *covariant derivative along* γ is the unique $\mathbb{R}$-linear operator

$$\frac{D}{dt} : \Gamma(\gamma^* E) \to \Gamma(\gamma^* E), \quad s \mapsto \frac{Ds}{dt}$$

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Also denoted ∇.

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- Define $\frac{Ds}{dt} := (\gamma^*\nabla)_{\partial_t} s$.

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- This satisfies Leibniz rule and compatibility by properties of the pullback connection.

Existence and Uniqueness

Theorem

There exists a unique covariant derivative along γ satisfying (i) and (ii).

Idea of proof:

- Use the pullback connection $\gamma^*\nabla$ on γ^*E.
- Define $\frac{Ds}{dt} := (\gamma^*\nabla)_{\partial_t} s$.
- This satisfies Leibniz rule and compatibility by properties of the pullback connection.
- Uniqueness follows because sections of the form $\sigma \circ \gamma$ generate $\Gamma(\gamma^*E)$ over $C^\infty(I)$.

Chain Rule

Lemma

Let $\phi : J \to I$ be a smooth map. Then

$$\frac{D}{dt}(s \circ \phi) = \left(\frac{Ds}{dt} \circ \phi\right) \frac{d\phi}{dt}.$$

Product Rule for Tensors

Suppose ∇ is an affine connection on TM, extended to tensor bundles.

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Lemma

Let T be a $(0, s)$-tensor field along γ with vector fields $X_1, \ldots, X_s$ along γ. Then

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Proof: Rewrite the equation:

$$\begin{aligned} &((\gamma^* \nabla)_{\partial_t} T)(X_1, \ldots, X_s) \\ &\qquad = \partial_t (T(X_1, \ldots, X_s)) - \sum_{i=1}^{s} T(X_1, \ldots, (\gamma^* \nabla)_{\partial_t} X_i, \ldots, X_s). \end{aligned}$$

Local Expression for Covariant Derivative

Let $(U, x^1, \ldots, x^n)$ be a chart on M and $(e_1, \ldots, e_r)$ a local frame of E.

$$s(t) = \sum_{j=1}^{r} s^j(t) e_j|_{\gamma(t)}, \quad \nabla_{\partial_i} e_j = \sum_{k=1}^{r} \Gamma_{ij}^k e_k$$

where Γ_{ij}^k are the connection coefficients.

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In local coordinates, the covariant derivative of s along γ is

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Matrix form: If $s(t)$ is view as a column vector,

$$\frac{Ds}{dt} = \frac{ds}{dt} + As,$$

where $A(t)$ has entries $A_j^k(t) = \sum_{i=1}^{n} \Gamma_{ij}^k|_{\gamma(t)} \dot{\gamma}^i(t)$.

Parallel Transport

Parallel Sections

Definition

A section $s \in \Gamma(\gamma^* E)$ along a smooth curve $\gamma : I \to M$ is called *parallel* if

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Proof: Consider a reparametrized curve $\tilde{\gamma} = \gamma \circ \phi$ with $\phi : J \to I$, and define $\tilde{s} = s \circ \phi \in \Gamma(\tilde{\gamma}^* E)$. Using the chain rule:

$$\frac{D\tilde{s}}{dt} = \left(\frac{Ds}{dt} \circ \phi \right) \frac{d\phi}{dt}.$$

so vanishing is equivalent.

Existence and Uniqueness of Parallel Sections

Theorem

Given $e_0 \in E_{\gamma(t_0)}$, there exists a unique parallel section $s \in \Gamma(\gamma^* E)$ with $s(t_0) = e_0$.

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with smooth matrix $A(t)$.Standard ODE theory guarantees a unique local solution. Glue local solutions using uniqueness.

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For a smooth curve $\gamma : [a, b] \to M$, the *parallel transport map* is

$$P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}, \quad e \mapsto s(b),$$

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Remark: For piecewise smooth curves, compose the parallel transports along each smooth segment:

$$P_\gamma = P_{\gamma|_{[t_{k-1},t_k]}} \circ \cdots \circ P_{\gamma|_{[t_0,t_1]}}.$$

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Holonomy of a Connection

Holonomy Group

Definition

The *holonomy group* of a connection ∇ at $p \in M$ is

$$\mathrm{Hol}_p(\nabla) = \{P_\gamma : E_p \to E_p \mid \gamma \text{ is a piecewise smooth loop based at } p\}.$$

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which shows the inclusion $\subseteq$. The reverse inclusion follows similarly.

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For a Riemannian metric g on M, the *Riemannian holonomy group* at $p \in M$ is

$$\mathrm{Hol}_p(g) = \mathrm{Hol}_p(\nabla),$$

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If M is orientable:

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Hence $t \mapsto g(V(t), W(t))$ is constant, so

$$g_p(v, w) = g_p(P_\gamma v, P_\gamma w).$$

Proof: Orientation Preservation (if M orientable)

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Thus $t \mapsto \omega(E_1(t), \ldots, E_n(t))$ is constant. Evaluating at $t = 0, 1$:

$$\omega_p(e_1, \ldots, e_n) = \omega_p(P_\gamma e_1, \ldots, P_\gamma e_n) \implies \det(P_\gamma) = 1.$$

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Questions?