Partie 1

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Section 2: Kähler Geometry and Holonomy

Our first goal is to define complex and almost-complex structures.

Let M be a real manifold of dimension $2n$.
An almost-complex structure J on M is a $(1,1)$ tensor J^a_b on M such that $J^a_b J^b_c = -\delta^a_c$.
This is equivalent to say that there exists J an endo on TM such that $J^2 = -\mathbb{1}_{TM}$.

The proof is as follows:

The $(1,1)$ tensor bundle is defined to be: $T^{(1,1)}M = TM \otimes T^*M$.

So $J \in \Gamma(T^{(1,1)}M)$.

Theorem: There is a natural iso: $\Phi : TM \otimes T^*M \longrightarrow \mathrm{Hom}(TM, TM)$

proof: On the fibers: $\Phi_p(v \otimes w)(x) = w(x)\, v$

Φ is well defined, since defined on pure tensors, so it lifts to $TM \otimes T^*M$.

Let $\{\partial/\partial x_i\}$ be a basis of TM, and $\{dx_i\}$ its dual basis. Then, $\{\partial/\partial x_i \otimes dx_j\}$ is a basis of $TM \otimes T^*M$.

Applying Φ yields: $\Phi(\partial/\partial x_i \otimes dx_j)(\partial/\partial x_k) = dx_j(\partial/\partial x_k)\, \partial/\partial x_i = \delta^j_k\, \partial/\partial x_i$

So $\Phi(\{\partial/\partial x_i \otimes dx_j\})$ is a linear map sending $\partial/\partial x_i \mapsto \partial/\partial x_i$, $\partial/\partial x_k \mapsto 0$ $\forall k$, and these are a basis of TM.

(We can verify, sometimes by hand, that, as usual.)

Corollary: Φ induces an iso. On sections: $\tilde{\Phi} : \Gamma(TM \otimes T^*M) \longrightarrow \Gamma(\mathrm{Hom}(TM, TM))$.

Also, as proven by Paul Bouchaud on the previous sessions: (Thm 1.5)

$$\Gamma(TM \otimes T^*M) \xrightarrow[\psi]{\sim} \mathrm{Sol}_{C^\infty(M)}(\Gamma(T^*M) \times \Gamma(TM), C^\infty(M)).$$

Combining these results, we get three possible descriptions of $(1,1)$ tensors:

$$J \in \Gamma(TM \otimes T^*M) \Leftrightarrow \underset{(\forall x \in \Phi)}{\tilde{J} : TM \longrightarrow TM} \Leftrightarrow \underset{(\text{Thm } 1.5)}{\hat{J} : \Gamma(T^*M) \times \Gamma(TM) \longrightarrow C^\infty(M)}$$

Construction 0:

Given $J \in \Gamma(TM \otimes T^*M)$, $\tilde{J} := \Phi(J) : M \longrightarrow \mathrm{Hom}(TM, TM)$, $p \mapsto \Phi_p : T_pM \to T_pM$.

Let $\hat{J} : TM \to TM$; $\hat{J}(v) = \tilde{J}(\pi(v))(v)$.

So, if, by definition, given a vector field V^a, $(JV)^b = J^b_a V^a$, then is equivalent of saying that $(JV)^b = \hat{J}(V)$ for $\hat{J}$ seen as an endo on TM.

Also: $(J \circ J)^c_a = J^c_b J^b_a = -\delta^c_a$ is equivalent to $\hat{J}^2 = -\mathbb{1}_{TM}$.

since for any vector field V^a:

$$-V^a = (J \circ J)^c_a V^a = J^c_b J^b_a V^a = J^c_b (JV)^b = \hat{J}(\hat{J}(V))$$

So $\hat{J}(\hat{J}(V)) = -V \quad \forall V \in \Gamma(TM)$

Now, if we fix $p \in M$ and $v \in T_pM$, there exists a vector field $X \in \Gamma(TM)$ such that $X(p) = v$. (Since we can define $X = Y(\exists v)$ for $\exists$ bump function, Y local field around p s.t. $Y(p) = v$.)

Given an almost-complex structure (M, J), we can define a complex structure on each tangent space T_pM.
In this context, we define the $(3,2)$-Tensor known as the *Nijenhuis tensor*:

$$N_J[X,Y] = [X,Y] + J([JX, Y] + [X, JY]) - [JX, JY]$$

The almost-complex structure J is said to be a complex structure if $N \equiv 0$.

Def: A "complex manifold" (M, J) is a manifold M, endowed with a complex structure J.

Question: "Complex Manifold" $\Leftrightarrow$ complex manifold ?

Lemma:
When given a complex manifold (M), we can define an endomorph $J : TM \to TM$; $J^2 = -\mathbb{1}_{TM}$.

Proof:
Sure, given $p \in M$, U coordinate based around p by a complex chart $\phi : U \to \mathbb{C}^n$, defined: $\mathbb{R}^{2n} \xrightarrow{\sim} \mathbb{C}^n$, now then, we can identify $T_pU \cong U \times \mathbb{C}^n$,
on which: $\sigma = (\mathbb{1}, i) : U \times \mathbb{C}^n \longrightarrow U \times \mathbb{C}^n$. (so that $\sigma^2 = (\mathbb{1}, -\mathbb{1})$.)
these collection of σ, glues together to $J : TM \to TM$: $J^2 = -\mathbb{1}$. $\square$

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Moreover, we have:

Lemma: For f holo on (X, J) complex manifold with J constructed as in the previous lemma, $N_J(\mathbb{1}, W)(f) = 0$ for all holo. map f.

Remark:
$$\begin{cases} df([V,W]) = [V,W](f) \text{ (by definition of the LB).} \\ df(JX) = i\, df(X) \quad (f \text{ holo, so its LB is } \mathbb{C}\text{-linear}). \\ \Leftrightarrow \\ J(V(f)) = i\, (V(f)) \end{cases}$$

Use: $df([W(V,W)]) = df([J_V, JW]) - df(J[JV, W]) - df(J[V, JW]) - df([V,W])$
$= [J_V, JW](f) - J[JV, W](f) - J[V, JW](f) - [V,W](f)$
$= [JV, JW](f) - i[JV, W](f) - i[V, JW](f) - [V,W](f)$ (1)

But: $[JV, JW](f) = JV(JW(f)) - JW(JV(f)) = JV(iW(f)) - JW(iV(f))$
$= iJV(W(f)) - iJW(V(f))$ (2)

Expanding (1) and using (2), the expression cancel out, given $N_J[V,W](f) = 0$ $\square$

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Corollary: Given (X, J) a complex manifold, with J induced by the cpx atlas, $N_J \equiv 0$.

This follows from the fact that given complex coordinates around a point $(z_1, \dots, z_n)$, $dz_i = dx_i + i\, dy_i$, with $\{\|dx_i\|, \|dy_i\|\}$ spanning T^*M.
Since $dz_i \wedge N_J = 0$ (as seen previously), consequently, $z_i = dx_i + i\, dy_i$, we get: $N_J \equiv 0$. $\square$

We have proved that (M, J) complex manifold $\Rightarrow$ (M, J) "complex manifold".
The converse direction is known as the modern formulation of the Newlander-Nirenberg Theorem.

Rk: Somehow $N_J \equiv 0$ allows to construct holomorphic functions.

The idea is the following, let's imagine the "Cauchy Riemann" equations on a function $f : M \to \mathbb{C}$.
When M is a 2D real "complex manifold": $f(x_1, \dots, x_{2n}) = u(x_1, \dots, x_{2n}) + i\, v(x_1, \dots, x_{2n})$.
(I'm about $f = u + iv$). Then the Cauchy-Riemann eq. for J would be:

$$\begin{cases} du = J(dv) \\ dv = -J(du) \end{cases}$$

Locally: $du = J(dv) \Rightarrow \sum \partial u/\partial x_i \, dx_i = J(\sum \partial v/\partial x_i \, dx_i) = \sum \partial v/\partial x_i \, J(dx_i)$

$\Rightarrow \partial u/\partial x_i = \partial v/\partial x_i$ when $J(dx_i) = dx_i$.

So we get a equation for u

and: $dv = -J(du) \Rightarrow \partial v/\partial x_i = -\partial u/\partial x_i$ when $J(dx_i) = dx_i$

So we would have to solve 2n equations for 2 functions. As it turns out, the fact that $N_J \equiv 0$ implies that these conditions are independent, so we can find real u, v.

2.2 Kähler manifolds.

Let (M, J) be a complex manifold, g a Riemann metric on M. We say that M is Hermitian if $g(V, W) = g(JV, JW)$ for all vector fields.
When g is Hermitian we define the Hermitian form ω of g by: $\omega(V, W) = g(JV, W)$.

Remark: We can reconstruct the Hermitian metric g using the Hermitian form since:

$$\omega(V, JW) = g(JV, JW) = g(V, W)$$

We say that g is Kähler if it's hermitian for (M, J) a complex manifold and one of the three equivalent definitions holds, for the Levi-Civita connection ∇:

1) $d\omega = 0 \Leftrightarrow \nabla J = 0 \Leftrightarrow \nabla \omega = 0$ (The equivalence is not obvious, its a theorem).

Proposition: A metric g on a $2n$-manifold M is Kähler with respect to some complex structure J on M $\Leftrightarrow$ $\mathrm{Hol}(g) \subset U(n) \subset O(2n)$

Question: How do we construct Kähler metrics? How abundant are they?

2.3 Kähler potentials

Let (M,J) be a complex manifold. For each Kähler metric g, we have an associated $(1,1)$-form ω such that $d\omega = 0$. Conversely, given a closed $(1,1)$ form ω, we can construct a Kähler metric g iff $\omega(v,Jv) > 0$. $\quad (\omega, v \neq 0)$

<u>proof</u> $\Rightarrow$ If $g(v,w) = \omega(v,Jw)$ Then $g(Jv,Jw) = \omega(Jv,J^2w) \overset{\omega \text{ bilinear}}{=} \omega(v,-w) = g(v,w)$

Moreover $g(v,v) = \omega(v,Jv) = \omega(Jv,J^2v) = \omega(Jv,-v) = \omega(v,Jv)$

$= g(v,v) > 0$ + bilinear.

$= g(v,w) = g(w,v)$, so it's symmetric.

Finally, $\eta_J = \eta\omega$ ω by hypothesis. $\square$

Then, ω is closed, since we can locally construct closed $(1,1)$ form. Let $\beta \in M \to M$ be a smooth function on base $\mathcal{U}$ $\mathcal{U}^{-1}(\mathcal{U}) \to \mathcal{U}^{-1}(\mathcal{U})$, as $\partial\bar\partial\beta = \alpha$ $(1,1)$ form. It is closed since

$$d(\partial\bar\partial\beta) = (\partial+\bar\partial)(\partial\bar\partial\beta) = \partial^2\bar\partial\beta + \bar\partial\partial\bar\partial\beta = 0 - \partial\bar\partial^2\beta = 0$$

Locally, the converse also holds

<u>Proposition</u> Every closed real 2-form ω can be locally written as a Kähler potential. i.e. form of the form $i\partial\bar\partial$.

<u>Proof</u>:

$d\omega = 0$ Using the real version of the Poincaré lemma for d, at every real $p \in M$, there exists a ball U so that $\omega|_U = d\alpha$ for $\alpha \in \Gamma(U, \Omega^1_U)$

Therefore $\omega = \partial\alpha + \bar\partial\alpha - \partial\bar\alpha^{1,0} + \bar\partial\bar\alpha^{0,1} = \bar\partial\bar\alpha^{0,1} + \bar\partial\bar\alpha^{0,1}$

Since $\bar\partial\alpha^{1,0} = 0$, by the $\bar\partial$-poincaré lemma $\alpha^{1,0} = \bar\partial\beta$ for $\beta \in \Gamma(U, \mathcal{C}^0_U)$

We $\mathcal{K}(\mathbb{C})$ is a bound. Therefore, we need to $\alpha^{1,0} = \bar\partial\beta = \overline{\partial^{1,0}}$, so $\alpha^{1,0} = \bar\partial\bar\beta$, and finally

$$\omega|_U = \bar\partial\bar\beta + \bar\partial\bar\beta - \partial\bar\beta - \partial\bar\beta$$
$$= i\partial\bar\partial(a\beta) \quad \square$$

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I summary, given a 2-form ω, we can locally pick it as a Kähler potential. however this is not true globally for example, if M is compact.

<u>Proof</u> Let ω compact Kähler manifold (M,ω), for simplicity $c_1(\omega, M)$, where $\eta = \deg M$. The Bellman cohomology groups $H^{p,q}(M,\mathbb{C})$ are non-trivial.

<u>Proof</u> $Vol(M) = \int_M \omega^n$ if $\omega^n = d\alpha$ for some α, then

$$n!\,Vol(M) = \int_M \omega^{n-1}\wedge\omega^n = \int_M \omega^{n-1}\wedge d\alpha = \int_M d(\omega^{n-1}\wedge\alpha) = 0$$
$$d\omega = 0 \qquad (\text{Stokes})$$

But by Hodge's theorem $H^{1,1}(M,\mathbb{C}) \simeq \bigoplus_{i+j=2} H^{i,j}(M,\mathbb{C})$

in particular, $\omega^2 \in H^{2,2}(M,\mathbb{C})$ is non-trivial, so it cannot exact. ω is not trivial

In particular, $\omega \in H^{1,1}(M,\mathbb{R}) \subset H^{1,1}(M,\mathbb{C})$ is non-trivial, so it cannot exact. $\omega \neq \partial\bar\partial\beta$ We will now state another version of the Poincaré lemma

<u>Lemma</u> ($\partial\bar\partial$-Poincaré lemma) Let ω be a $(1,1)$ form such that $d\omega = 0$ and $\int \omega \wedge \omega \in H^4(M,\mathbb{R})$. Then there is a real function ϕ so that $\omega = i\partial\bar\partial\phi$

(The proof can be found in Huybrechts at Alg. Geo. by Griffiths and Harris on page 357)

<u>Corollary</u> Let $\int \omega \wedge \omega \in H^4(M,\mathbb{R})$ be two Kähler forms on M. Then, there exist $\phi \in \mathcal{C}^\infty(M)$ such that $\omega = \omega' + i\partial\bar\partial\phi$. The latter is always up to addition of a constant. $\square$

We can use Kähler potentials to construct Kähler metrics. Let (M,η,ω) be a compact Kähler manifold. Then, for every $\phi \to M \to \mathbb{C}^\infty$ function, the form $\omega + i\partial\bar\partial\phi$ is closed, type $(1,1)$ form on M.

If $\omega' = \omega + i\partial\bar\partial\phi$ are positive definite, then the Lemma proven of the beginning of the chapter, ω' would yield a Kähler metric g'.

Since M is compact, the space $SM = \{\forall \in TM : \eta(v,v) = 1\}$ is compact. Now, since ω is positive definite, the function $SM \xrightarrow{\omega(v,Jv)/\eta(v,v)} \mathbb{R}_{>0} \in \mathbb{R}$ is well defined, therefore it attains a minimum $c > 0$, for that $\omega(v,Jv) > c \quad \forall v \in SM$.

Similarly, $\exists\, c' > 0$ so that $\|i\partial\bar\partial\phi(v,Jv)\| \leq c'\|i\partial\bar\partial\phi\|_{\mathcal{C}^0}$

$$\leq c'|\phi|_{\mathcal{C}^2} \text{ for some } C = c'$$
$$= C\|\phi\|_{\mathcal{C}^2}$$

So, $\omega(v,Jv) = \omega(v,Jv) + i(\partial\bar\partial\phi)(v,Jv) \geq C - C\|\phi\|_{\mathcal{C}^2}$, if we take ϕ so that $\|\phi\|_{\mathcal{C}^2} < \frac{c}{2C}$, we get

$$\omega(v,Jv) \geq c/2 \quad \forall v \in SM.$$
$$\Rightarrow \omega(X,JX) \geq c/2\,|X|^2 \quad \text{for } X \in TM,$$

So $\omega(\phi) = \omega + i\partial\bar\partial\phi$ yields smooth Kähler metric. $\square$

<u>Corollary</u>: If M admits one Kähler metric, then it admits an infinite number of them.

2.4 Ricci curvature

Recall the definition of the curvature tensor on a Riemannian manifold for the Levi-Civita connection associated to the metric g. Given vector fields $X, Y \in \mathfrak{X}(M)$,

$$R(X,Y): \mathfrak{X}(M) \to \mathfrak{X}(M)$$
$$Z \mapsto \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]} Z.$$

Using the Riemannian curvature, we can define the Ricci curvature.

Ricci curvature and the Ricci form

Let M be a Riemannian Manifold. For a point $p \in M$, choose a unitary vector $v \in T_pM$. If $\dim(M) = n$, then dim $(\ker(v^\perp)^\perp) = n-1$, for T_pM.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of such $(v^\perp)^\perp$. We define the Ricci curvature of M at p in the direction of v as:

$$Ric_p(v) = \frac{1}{n-1}\sum_{i=1}^{n} \langle R(e_i,v)v, e_i\rangle$$

We can extend the definition to any $v \in T_pM$.

$$Ric_p(v) = |v|^2\, Ric(\frac{v}{|v|})$$

Since $Ric_p: T_pM \to \mathbb{R}$ is symmetric from the metric $\langle\,,\rangle$ bilinear symmetric form

$Ric_p(v,w): T_pM^2 \to \mathbb{R}$ such that $Ric_p(v,w) = Ric_p(w,v)$

The definition is very geometric and intuitive, but can be a bit cumbersome since it depends on the choice of a basis.

However, it can be shown that

$$Ric_p(v,w) = Tr_v(\mathfrak{X} \mapsto R(\mathfrak{X},v)w)$$

where Tr_v is the trace with respect $(\mathfrak{X} \mapsto R(\mathfrak{X},v)w)$, which has not depend on the basis. We can define the Ricci curvature tensor as:

$$Ric \in \Gamma(T^*M \otimes T^*M) \text{ for } X(M) \times X(M) \to \mathcal{C}^\infty(M) \quad \text{tensor not } \mathcal{C}^\infty(M)$$

$$Ric(X,Y) = Tr_X(\mathfrak{Z} \mapsto R(\mathfrak{Z},X)Y)$$

In the Kähler setting, define the Ricci form for p as

$$\rho(X,Y) = Ric(\mathfrak{Z}X,Y)$$

Let R_{ij} be the components of the Ricci tensor. Taking complex coordinates, it can be shown that locally, ρ can be written as

$$\rho = -R_{ij}\, dz^i \wedge d\bar z^j.$$

where R_{ij} is the component of R in the complex coordinates $dz, d\bar z$ of $\mathbb{C}$.

In particular, this shows that $\rho \in$ real $(1,1)$ form, $\rho \subset \Omega^{1,1}(M,\mathbb{R})$.

In complex coordinates, $R_{ij} = -\partial_i\bar\partial_j \log\det(g_{k\bar l})$

so that $\rho = -i\partial_i\bar\partial_j \log\det(g_{k\bar l})\, dz \wedge d\bar z$.

For the readers who are familiar with characteristic classes theory, it can be shown that

$$[c_1] = 2\pi\, c_1(M) \quad \text{in } H^2(M,\mathbb{Z}),$$

where c_1 is the first chern class of M.

2.5 Calabi-Yau Manifolds

<u>Def</u>: Let $n \geq 3$. A Calabi-Yau m-fold is a quadruple (M,Ω,η,ω) such that (M,η) is a compact m-dimensional complex manifold, η is a Kähler metric on (M,η) with holonomy group $Hol(\eta) = SU(m)$ and Ω is a constant $(m,0)$ form on M called the holomorphic volume form such satisfy

$$\omega^m/m! = i^{m^2} (\sqrt{\eta})^n \wedge \Omega\bar\Omega; \quad (1)$$

where ω is the Kähler form of η.

<u>Theorem</u> The condition (1) is equivalent to asking a Ricci-flat metric where M is compact Kähler manifold.

(This can be proved after proving the calabi-Yau conjecture, which we will do in next version)

The condition on the existence of Ω is not necessary, although a second definition

<u>Proposition</u> Let (M,Ω,η) be a compact Kähler manifold with $Hol(\eta) = SU(m)$.

Then, M admits a holomorphic volume form Ω satisfying (1), which is unique up to change of phase $\Omega \to e^{i\theta}\Omega$.

<u>proof</u>

Let (M,Ω,η) be a compact Kähler manifold with $Hol(\eta) = SU(m)$

Let $\eta \in Hol_x(M)$. $\forall \xi \in T_xM \to T_xM$ $\forall \xi$ is by definition unitary, ξ-linear and has determinant 1.

Therefore, it has flat of the shape $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$ with λ_i ξ ξ

$\xi_1 = dz_1^1 \dots dz_m^m$, $\omega_1 = \xi_1(dz_1 \dots dz_m)$, $\omega_2 = \xi_2(dz_1 \dots dz_m)$

so that $\begin{cases} \xi_1(\xi_1(v_1,v_2)) = (\lambda_1\lambda_2)(v_1 \dots v_m) = \xi_1(\omega_1(v_1 \dots v_m)) \\ \omega_1(\xi_1(v_1,v_2)) = \xi_1\,(\lambda_1(v_1,v_2) \dots \lambda_m(v_1,v_2)) = \xi_1(\omega_1(v_1 \dots v_m)) \\ \xi_2^2(dz_1 \dots) = \xi_2^2(\omega_1)\wedge \dots \wedge\omega_m^2 = \lambda_1\lambda_2(dz_1 \dots \wedge dz_m) \end{cases}$

Since $\xi_1^2\,\Omega = \lambda_1\lambda_2(\xi_1)\,\Omega = \Omega$

Let θ be a linear connection between $(T_x M, \eta)$ and $(\mathbb{C}^n, \eta_0)$ is flat

$\eta(\xi_1 v) = \eta_0(\xi_1 v, \xi_1 v)$. We define $\Omega := \theta^{-1}\Omega_0$.

Since Ω is a constant, and is a section given into locally the metric

respondes Ω is also unique, and therefore, the existence under $Hol(\eta)$ will follow for (η, ω, Ω).

To illustrate, they are well, let the computation. $\square$

Since Ω is a constant, we get that $\theta^*\Omega_0 = \eta_1(\xi)\,\Omega_0 = \eta(\xi, -)$ $= \eta$.

Let now $A = \Omega \wedge \bar\Omega \in \Omega^{2m}(M)$.

$$\nabla(\theta_1^*\Omega_0^{2m}) = \theta_1^*\nabla\Omega^{2m} = (\lambda_1\omega)^{2m} = \theta^*\Omega^{2m} = \theta^*\eta_0 = \eta$$

Therefore $\nabla\eta = \eta$

Therefore, we have shown that the becomes η_1 and Ω are parallelism, consider by theorem 2.3, we get a global tensor A such that $\nabla\Omega = 0$.

Remark: we already know that $\nabla\eta_1 = \nabla\omega$.

Let now $A = \Omega^{2m}/m!$ and $B = (-1)^{m(m-1)/2}(\sqrt{\eta})^n \wedge \Omega \wedge \bar\Omega$.

Then, $\nabla(A-B) = 0$, since $\nabla(A) = 0 = \nabla B$

Now $(A-B)^2 = 0$, by the Lemma proven we get that

$$\omega/m! = (-1)^{m(m-1)/2}(\sqrt{\eta})^n \wedge \Omega \wedge \bar\Omega$$

on all M. $\square$

<u>Lemma</u>: Let ∇ $\Gamma(T^{(r,s)}M)$ be a tensor field such that $\nabla\nabla = 0$.

Let $\gamma: [a,b] \to M$ be a smooth curve.

Then $P_\gamma(T_{\gamma(a)}) = T_{\gamma(b)}$

<u>Corollary</u> (Uniqueness of parallel tensor given initial value)

If $T, S \in \Gamma(T^{(r,s)}M)$ satisfy $\nabla T = \nabla S = 0$, and $T_p = S_p$ for some $p \in M$, then $T = S$ on all M.

<u>proof</u> Let M a connected, for any $q \in M$, there exist a smooth path from p to q.

applying the Lemma $T_q = P_\gamma(T_p) = P_\gamma(S_p) = S_q$