K3 Surfaces

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1 Introduction

If X is a compact Riemann Surface of gender g. Let's suppose its canonical bundle K_X is trivial *ie* $K_X \simeq \mathcal{O}_X$, in particular if K is a canonical divisor then we have $\deg(K) = 0$. But we know by the Riemann-Roch formula that $\deg(\mathrm{K}) = 2g - 2$, so we deduce that $g = 1$ and that X is an elliptic curve. Notice that the condition for a complex manifold of being Calabi-Yau implies that the canonical line bundle is trivial, we have classified all Calabi-Yau manifold of dimension 1.

We can now ask the same question in bigger dimension : *What are the compact complex manifold with trivial canonical bundle ?*

Definition 1.1: In dimension 2, Weil has noticed that the class of such surface S (compact with $K_S \simeq \mathcal{O}_S$) with $H^1(S, \mathcal{O}_S) = 0$ has a rich and rigid geometry. He named these surface the K3 surface "in honor of Kummer, Kähler, Kodaira and of the beautiful mountain K2 in Kashmir."

2 Examples

— We will first search examples in the projective spaces : Take S to be a smooth quartic of $\mathbb{P}^3_{\mathbb{C}}$ defined by the homogenous polynomial f. Seeing S as a divisor we deduce that the ideal sheaf of $S \subset \mathbb{P}^3_{\mathbb{C}}$ is $\mathcal{O}(-S) \subset \mathcal{O}$. The multiplication by f induces an injection $\mathcal{O}(-4) \to \mathcal{O}$ and the image is the ideal sheaf of S thus we get an short exact sequence :

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_S \to 0$$

Using Cech cohomology we can prove that $H^1(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}) = H^2(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}(-4)) = 0$ and the long exact sequence in cohomology induced by the short one shows that $H^1(S, \mathcal{O}_S) = 0$.
In addition we know from the adjunction formula and corollary 2.4.9 from [Huya] that $K_S \simeq \mathcal{O}(\deg(S) - 3 - 1)|_S = \mathcal{O}_S$. We have our first example of K3 surface, an interesting example is when we take S to be the Fermat quartic defined by the polynomial $x_0^4 + x_1^4 + x_2^4 + x_3^4$.

— In the previous example, we have seen that the condition $K_S \simeq \mathcal{O}_S$ for $S \subset \mathbb{P}^3_{\mathbb{C}}$ forces $\deg(S)$ to be 4. It leads us to the same question in $\mathbb{P}^n_{\mathbb{C}}$ when $n \geq 3$. We search for a surface, then the examples must be (complete smooth) intersections of type $(d_1, ..., d_{n-2})$ *ie* the zeros loci of $n-2$ polynomials of degree $d_1 \leq d_2 \leq \cdots \leq d_{n-2}$. The canonical bundle may be compute with the adjunction formula :
Denote by $f_1, ..., f_{n-2}$ the polynomials defining S. By hypothesis the $\mathrm{Jac}_x(f_1, \cdots f_{n-2})$ is surjective on each point, so TS is given by

$$TS = \bigcap_{i=1}^{n-2} Ker(df_i)$$

Where the intersection is taken fibre by fibre in $T\mathbb{P}^n_{\mathbb{C}}$. Hence,

$$(\mathcal{N}_{S/\mathbb{P}^n_{\mathbb{C}}}) = T\mathbb{P}^n_{\mathbb{C}}|_S / \left(\bigcap_{i=1}^{n-2} Ker(df_i)|_S \right) = \bigoplus_{i=1}^{n-2} T\mathbb{P}^n_{\mathbb{C}}/Ker(df_i)|_S = \bigoplus_{i=1}^{n-2} T\mathbb{P}^n_{\mathbb{C}}/TV(f_i)|_S = \bigoplus_{i=1}^{n-2} \mathcal{N}_{V(f_i)/\mathbb{P}^n_{\mathbb{C}}}|_S$$

But the argument given in the first example show that $\mathcal{N}_{V(f_i)/\mathbb{P}^n_{\mathbb{C}}} = \mathcal{O}(d_i)|_{V(f_i)}$, thus $\mathcal{N}_{S/\mathbb{P}^n_{\mathbb{C}}} = \bigoplus \mathcal{N}_{V(f_i)/\mathbb{P}^n_{\mathbb{C}}}|_S = \bigoplus \mathcal{O}(d_i)|_S$. Therefore, we have

$$K_S \simeq K_{\mathbb{P}^n_{\mathbb{C}}}\big|_S \otimes \det(\mathcal{N}_{S/\mathbb{P}^n_{\mathbb{C}}}) \simeq \left(\mathcal{O}(-n-1) \otimes \bigotimes_{i=1}^{n-2} \mathcal{O}(d_i)\right)|_S \simeq \mathcal{O}(d_1 + \cdots + d_{n-2} - n - 1)$$

To have a K3 surface, we must have $d_1 + \cdots + d_{n-2} = n+1$ but we can suppose that $d_1 \geq 2$ otherwise $V(f_1) \simeq \mathbb{P}^{n-1}_{\mathbb{C}}$ and thus S can be identify as a (complete smooth) intersections in $\mathbb{P}^{n-1}_{\mathbb{C}}$. Then we have $2(n-2) \leq d_1 + \cdots + d_{n-2} = n+1$ and so $n \leq 5$:

- If $n = 3$: this is the case of the previous examples with one polynomial of degree 4.
- If $n = 4$: the only possibility is the intersection of a quadric and a cubic *ie* $d_1 = 2$ and $d_2 = 3$.
- If $n = 5$: the only possibility is the intersections of 3 quadric *ie* $d_1 = d_2 = d_3 = 2$.

One can compute $H^1(S, \mathcal{O}_S)$ for the previous examples and show that it is 0. (I DON'T KNOW HOW TO DO IT !)

- An important example is the Kummer surface but we will not describe the construction of such a surface ; to avoid upsetting the reader, I have included some beautiful photos below :

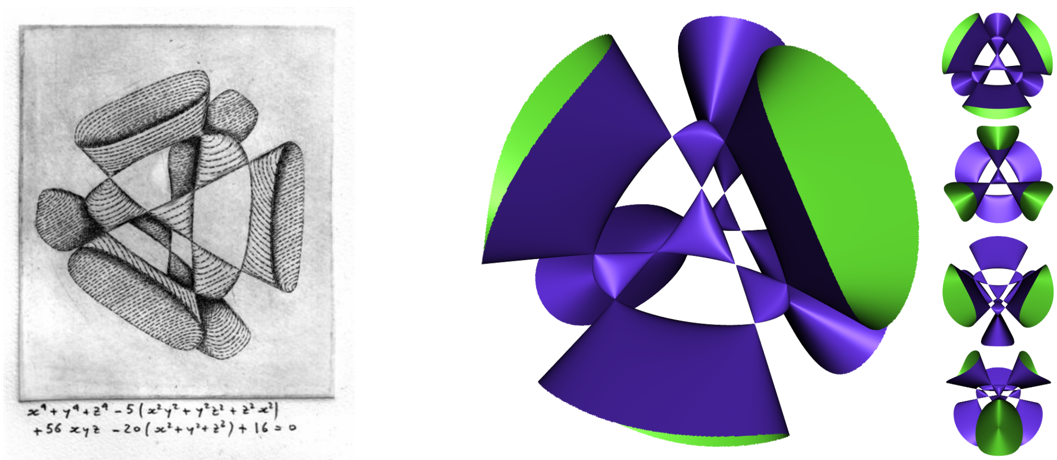

FIGURE 1 – The real points of the Kummer surface

3 Topological property and the K3 lattice

3.1 Complex Surface theory

Our goal now is to state a Riemann-Roch theorem for complex surface. In order to do such a thing, we need to introduce the intersection form of two line bundle.

Definition 3.1: Let L_1, L_2 be two line bundle on S, the intersection form $L_1 \cdot L_2$ is the integer defined by :

$$L_1 \cdot L_2 := \chi(S, \mathcal{O}_S) - \chi(S, L_1^\vee) - \chi(S, L_2^\vee) + \chi(S, L_1^\vee \otimes L_2^\vee) \in \mathbb{Z}$$

Where $\chi(S, \mathcal{F})$ denote the Euler characteristic of the sheaf $\mathcal{F}$. We will use the notation L^2 for $L \cdot L$. There exists a geometric interpretation of this number that will be not detailed here.

Theorem 3.2: The intersection form is symetric and bilinear on $Pic(S)$.
Demonstration: Admit, [Deb20] refers to [Bea] theorem I.4 and so do I.

One can also show that $L_1 \cdot L_2 = c_1(L_1) \smile c_1(L_2)$. This previous relation is well defined as $c_1(L_1) \smile c_1(L_2)$ lies in $H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$ thanks to the Poincaré duality.

Theorem 3.3: Riemann-Roch for complex surfaces Let L be a line bundle on a complex surface S. We have

$$\chi(S, L) = \chi(S, \mathcal{O}_S) + \frac{1}{2}\left(L^2 - L \cdot K_S\right)$$

Demonstration: If we compute $L^\vee \cdot (L \otimes K_S^\vee)$ within to different ways :

$$L^\vee \cdot (L \otimes K_S^\vee) = L^2 - L \cdot K_S \quad (1)$$

Using the bilinearity of the intersection form and the fact that $L \cdot \mathcal{O}_S = 0$. In addition we have

$$L^\vee \cdot (L \otimes K_S^\vee) = \chi(S, \mathcal{O}_S) - \chi(S, L) - \chi(S, L^\vee \otimes K_S) - \chi(S, L \otimes L^\vee \otimes K_S)$$

But Serre Duality implies that $\chi(S, \mathcal{O}_S) = \chi(S, K_S)$ and $\chi(S, L) = \chi(S, L^\vee \otimes K_S)$ (I did the calculation trust me). So we have

$$L^\vee \cdot (L \otimes K_S^\vee) = 2\chi(S, \mathcal{O}_S) - 2\chi(S, L) \quad (2)$$

The equation (1) and (2) give the formula. □

3.2 The topology of K3 surfaces

Firstly, there is a theorem in algebraic topology that assert that the homeomorphism type of a real topological manifold of dimension 4 is completely determined by the "structure" of its H_2. Thus, we will study the integer cohomology of a K3 surface.

One can note that the Serre Duality has a simple expression for K3 surfaces :
Theorem 3.4: Serre Duality for K3 Let $\mathcal{F}$ be a coherent sheave on S a K3 surface. We have :

$$H^k(S, \mathcal{F}) \simeq H^{4-k}(S, \mathcal{F}^\vee)^\vee$$

Demonstration: We must notice that $\mathcal{F}^\vee \otimes K_S \simeq \mathcal{F}^\vee \otimes \mathcal{O}_S \simeq \mathcal{F}^\vee$ as the canonical bundle is trivial, and apply the general Serre Duality. □
Note that the same argument works for Calabi-Yau manifold.

The Riemann-Roch theorem also have its own reformulation :
Theorem 3.5: Riemann-Roch for K3 Let L be a lien bundle on S a K3 surface. We have

$$\chi(S, L) = 2 + \frac{L^2}{2}$$

Demonstration: The Serre duality implies $\chi(S, \mathcal{O}_S) = 2$ as $\mathcal{O}_S^\vee = \mathcal{O}_S$ and the intersection form $L \cdot K_S = L \cdot \mathcal{O}_S = 0$ so we get the formula. □

Lemma 3.6: The Picard group $Pic(S)$ of a K3 surface S is torsion-free.
Demonstration: Let $L \in Pic(S)$ be a torsion element and fix a line bundle $M \in Pic(S)$, notice that $M \cdot \_ : Pic(S) \to \mathbb{Z}$ is a group homomorphism, but $\mathbb{Z}$ does not have any torsion element, therefore $M \cdot L = 0$ in particular $L^2 = 0$. In addition, Serre duality for K3 and Riemann-Roch implies

$$h^0(S, L) - h^1(S, L) + h^0(S, L^\vee) = 2$$

Thereby, either L or $L^\vee$ has a non trivial section s, lets suppose, without loss of generality that s is a section of L. But there exists m such that $L^{\otimes m} = \mathcal{O}_S$, so $s^{\otimes m}$ is a non trivial section and so $s^{\otimes m}$ does not vanish, otherwise $s^{\otimes m}$ would be 0. Thus, s does not vanish either, so L has a nowhere zero holomorphic section, this implies that $L \simeq \mathcal{O}_S$ is trivial. □

Theorem 3.7: The integer cohomology of a K3 surface S is :

$$H^0(S, \mathbb{Z}) = H^4(S, \mathbb{Z}) = \mathbb{Z} \text{ and } H^1(S, \mathbb{Z}) = H^3(S, \mathbb{Z}) = 0$$

We will talk about the $H^2(S, \mathbb{Z})$ just after.
Demonstration: The fact that $H^0(S, \mathbb{Z}) = H^4(S, \mathbb{Z}) = \mathbb{Z}$ is given by the Poincaré duality because we

always have $H^0(S,\mathbb{Z}) = H_0(S,\mathbb{Z})$.
After that, the Serre Duality implies that $H^1(S,\mathcal{O}_S) = H^3(S,\mathcal{O}_S) = 0$. Write the exponential sequence :

$$0 \to \underline{\mathbb{Z}}_S \to \mathcal{O}_S \to \mathcal{O}_S^\times \to 0$$

The long sequence induced in cohomology implies that $H^1(S,\mathbb{Z}) \to H^1(S,\mathcal{O}_S)$ is an injection and thus $H^1(S,\mathbb{Z}) = 0$. Moreover, the Poincaré duality gives that $H_3(S,\mathbb{Z}) = 0$ and so the universal coefficient theorem implies that $H^3(S,\mathbb{Z}) = H_3(S,\mathbb{Z})_{free} \oplus H_2(S,\mathbb{Z})_{tors} = H_2(S,\mathbb{Z})_{tors}$.
We also find $0 \to Pic(X) \to H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O}_S)$. We proved that $Pic(X)$ does not have any torsion elements but it is the same for $H^2(S,\mathcal{O}_C)$ as it is a complex vector space. So $H^2(S,\mathbb{Z})$ does not have any torsion elements either, therefore the Poincaré duality say that $H_2(S,\mathbb{Z})_{tors} = 0$. Thus $H^3(S,\mathbb{Z}) = 0$. □

–We have seen that $H^1(S,\mathbb{Z}) = 0$ but we can show using deformation theory that any K3 surface is simply connected (see thm 7.1.1 of [Huyb]. There also exists a proof of this using Kähler-Einstein metrics in [Nie].

We now have to describe the $H^2(S,\mathbb{Z})$ group. As we have to use lattice theory, we will not give any proofs.

Definition 3.8: Lattice We say that $(\Lambda, (\_,\_))$ is a lattice of rank r if Λ is a group isomorphic to $\mathbb{Z}^r$ and $(\_,\_)$ is a symmetric bilinear form on Λ.
– A morphism of lattice is a morphism of group that preserve the bilinear form.
– A lattice of rank r is completely determined by the given of a basis $(e_1, ..., e_r)$ and of the matrix $A = (a_{ij})$ of the form $(\_,\_)$ in this basis. The matrix A is defined by $(x,x) = \sum_{0 \le i,j \le r} a_{ij} \cdot x_i x_j$. The lattice is said to be unimodular if $\det(A) = \pm 1$.

Examples 3.9:

— The hyperbolic lattice U is defined the lattice of rank 2 with the bilinear form defined by the matrix :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

— The E_8-lattice is given by the matrix :

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

— With a given lattice $(\Lambda, (\_,\_)_\Lambda)$, you can always construct the m-twisted lattice $\Lambda(m)$ by defining $\Lambda(m) := (\Lambda, m \cdot (\_,\_)_\Lambda)$.

Theorem 3.10: The K3 lattice If S is a K3 surface then we have an isomorphism of lattice :

$$H^2(S,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

Where $H^2(S,\mathbb{Z})$ is endowed by the cup product symmetric bilinear form. The right hand side lattice is called the K3 lattice.
Demonstration: We can show that the hypothesis of the theorem 5 of chapter V of [Ser77] are verified and hence we have the result.

Finally, we give a description of the Dolbeault cohomology.
Theorem 3.11: Hodge Diamond of a K3 The hodge diamond of a K3 surface is of the form :

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h^{1,1} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Demonstration: Firstly, Hodge duality implies that $h^{2,2} = dimH^4(S,\mathbb{C}) = dimH^0(S,\mathbb{C}) = h^{0,0} = 1$. Also we have $H^0(S,\Omega_S^2) = H^0(S,K_S) \simeq H^0(S,\mathcal{O}_S) = \mathbb{C}$ by hypothesis, then $h^{2,0} = 1$ and by conjugation $h^{0,2} = 1$. So we have computed the extremity of the diamond.
We also know that $H^1(S,\mathcal{O}_S) = 0$ *ie* $h^{0,1} = 0$, thus Hodge duality and the conjugation give that $h^{0,1} = h^{1,0} = h^{2,1} = h^{1,2} = 0$. □

–Note that $h^{1,0} = 0$ means that there is no global non zero holomorphic 1-form and by duality no global non zero holomorphic vector field.

A simple calculation knowing the Hirzebruch-Riemann-Roch theorem gives us the remaining value : $h^{1,1} = 20$. The Hodge diamond become :

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

A final fact about the topology of K3 surfaces is that using the Calabi-Yau theorem, one can prove, as their first Chern class vanish, that there always exists Kähler metrics on K3 surfaces.

4 The period map of K3 surfaces

In this section I will "construct" the (naive) period map of K3 surface and state some results.

Lets denote by $(\Lambda_{K3}, (\_,\_))$ the K3 lattice. The complexifiation $\Lambda_{K3} \otimes \mathbb{C}$ is a complex vector spaces of dimension 22 that can be endowed with the $\mathbb{C}$-linear extension of $(\_,\_)$ also denoted by $(\_,\_)$. Then (x,x) can be considered as a homogeneous polynomial of degree 2.

Definition 4.1: The period domain of K3 surface is the following 20 dimensional complex manifold :

$$\mathcal{D} := \{x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}, (x,x) = 0,\ (x,\bar{x}) > 0\}$$

A marked K3 surface is a couple (S,α_S) of a K3 surface S and an isomorphism of lattices

$$\alpha_S : H^2(S,\mathbb{Z}) \to \Lambda_{K3}$$

Where the symmetric bilinear form of $H^2(S,\mathbb{Z})$ is the cup product of two 2-cycles that lies in $H^4(S,\mathbb{Z}) \simeq \mathbb{Z}$.
We also denote by $\alpha_S = \alpha_{S,\mathbb{C}} : H^2(S,\mathbb{C}) \to \Lambda_{K3} \otimes \mathbb{C}$ the complexification of α_S. Note that as α_S send the cup product of $H^2(S,\mathbb{Z})$ on the symmetric bilinear form $(\_,\_)$ of Λ_{K3}, $\alpha_{S,\mathbb{C}}$ verify :

$$\int_S \omega \wedge \eta = (\alpha_S(w), \alpha_S(\eta))$$

For such a surface, as K_S is trivial, there exists ω_S a nowhere zero holomorphic 2-form. Moreover, ω_S is such that

$$\int_S \omega_S \wedge \omega_S = 0 \quad \text{and} \int_S \omega_S \wedge \overline{\omega_S} > 0$$

Where the left hand side equality is given by the fact that $\omega_S \wedge \omega_S \in H^{4,0}(S) = 0$ and the right hand side inequality is given by the fact that it is the integral of the module of a non zero function and hence is strictly positive. Then we conclude that $[\alpha_S(\omega_S)] \in \mathcal{D}$, or equivalently, if we do not want to choose a ω_S, we say that $\alpha_S(H^{2,0}(S)) \in \mathcal{D}$ as $H^{2,0}(X) = \mathbb{C}\omega_S$.
In other words, for any marked K3 surface (S, α_S) we can associate the point $\alpha_S(H^{2,0}(S))$ in the period domain $\mathcal{D}$.

Propriété 4.2: For (S, α_S) a marked K3 surface and $\mathcal{X} \to (B, 0)$ a deformation on S with a contractible basis B (it also work with a simply connected basis). The following map, called the period map, is well defined :

$$\begin{array}{rccc} \mathcal{P}: & S & \to & \mathcal{D} \subset \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \\ & t & \mapsto & \alpha_S(H^{2,0}(X_t)) \end{array}$$

Demonstration: Firstly, it is not clear, a priori, that the fibres X_t are also K3 surfaces. It can be shown by a simple computation using the Hirzebruch-Riemann-Roch formula, we will admit it.
Secondly, as B is contractible we can suppose, since we could shrink B, that we have a marked isomorphism of lattices $\phi_t : H^2(X_t, \mathbb{Z}) \simeq H^2(X_0, \mathbb{Z})$ where $X_0 = S$ by definition. Thus, it gives a meaning to $\alpha_S(H^{2,0}(X_t))$: it is the composition $\alpha_S \circ \phi_t$ of this marked isomorphism and α_S. For the same reason,

$$0 = \int_{X_t} \omega_{X_t} \wedge \omega_{X_t} = \int_S \phi_t(\omega_{X_t}) \wedge \phi_t(\omega_{X_t}) = (\alpha_S \circ \phi_t(\omega_{X_t}), \alpha_S \circ \phi_t(\omega_{X_t}))$$

And

$$0 < \int_{X_t} \omega_{X_t} \wedge \overline{\omega_{X_t}} = \int_S \phi_t(\omega_{X_t}) \wedge \phi_t(\overline{\omega_{X_t}}) = (\alpha_S \circ \phi_t(\omega_{X_t}), \alpha_S \circ \phi_t(\overline{\omega_{X_t}})) = (\alpha_S \circ \phi_t(\omega_{X_t}), \overline{\alpha_S \circ \phi_t(\omega_{X_t})})$$

And then $\alpha_S(H^{2,0}(X_t)) \in \mathcal{D}$. $\square$

We will also admit that it exists a universal deformation $\mathrm{Def}(S)$ for all $K3$ surface S and this universal deformation is contractible.

Theorem 4.3: Local Torelli The period map $\mathcal{P} : \mathrm{Def}(S) \to \mathcal{D}$ is a local isomorphism at 0. *Demonstration:* Admit see [Huyb]. $\square$

–There also exists a global statement of the previous Torelli theorem that will not be discussed. You can find it in [Huyb].

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