Calabi-Yau threefolds and a brief introduction to Mirror Symmetry

Pietro Taddia

14 January 2026

Contents

Chapter 1

Calabi-Yau manifolds

1.1 Introduction

Definition 1. A *Calabi-Yau manifold* is a compact complex Kähler manifold X with trivial canonical bundle $K_X \cong \mathcal{O}_X$.

Remark 1. In the previous lectures we gave a different definition for a Calabi-Yau manifold. That definition implies that, if $\dim X = n$, than

$$\begin{aligned} h^{0,0} &= 1, \\ h^{n,0} &= 1, \\ h^{p,0} &= 0, \text{ for } p \neq 0, n. \end{aligned}$$

As a consequence of the following proposition and of Chow's theorem, which states that every complex submanifold of a projective space is an algebraic variety, we can study Calabi-Yau manifolds using the machinery of complex algebraic geometry

Proposition 1. *Let X be a compact complex Kähler manifold with $\dim X \geq 3$ and $h^{2,0} = 0$. Then X is projective.*

Proof. Let X be a manifold as above and let $\mathcal{K}_X$ be the Kähler cone of X. As X is a compact Kähler manifold, we know that $\mathcal{K}_X \subseteq H^{1,1}(X) \cap \mathrm{Im}\left(H^2(X,\mathbb{R}) \hookrightarrow H^2(X,\mathbb{C})\right)$ is an open, convex and non empty cone in a $\mathbb{R}$-vector space. For the Kodaira immersion theorem it suffices to prove that

$$\mathcal{K}_X \cap \mathrm{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\right) \neq \varnothing,$$

i.e. there exists $[\omega] \in \mathrm{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\right) \cap H^{1,1}(X)$, with ω Kähler form. Since $h^2(X,\mathcal{O}_X) = 0$, then $H^{2,0}(X) = H^{0,2}(X) = 0$. In particular

$$H^2(X,\mathbb{C}) = H^{1,1}(X).$$

Therefore it suffices to show that there exists $[\omega] \in \mathrm{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\right)$, with ω Kähler form. Moreover, we observe that $\mathrm{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})\right) \subseteq H^2(X,\mathbb{R})$ is a lattice.

We show that every non-empty, open, convex cone in an $\mathbb{R}$-vector space V of dimension n intersects every lattice $\Lambda \subseteq V$; indeed, up to an $\mathbb{R}$-linear isomorphism we can assume that $V = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$; it is trivial to observe that linear isomorphisms send non-empty, open, convex cones to non-empty, open, convex cones. Moreover, $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is dense, so every non-empty open set intersects $\mathbb{Q}^n$. In particular, if $C \subseteq \mathbb{R}^n$ is a non-empty open cone, then there exists $v \in C \cap \mathbb{Q}^n$. Up to multiplying by the product of the denominators of the components of v, we have $C \cap \mathbb{Z}^n \neq \varnothing$. This shows that

$$\mathcal{K}_X \cap \operatorname{Im}\left(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\right) \neq \varnothing.$$

$\square$

1.2 Why are Calabi-Yau manifolds interesting?

- For mathematicians: We have already discussed the Beauville-Bogomolov theorem

  Theorem 1. *Let X be a compact complex Kähler manifold with $c_1(X) = 0$. Then X has a finite unramified cover Y, with*

  $$Y \cong Z \times \prod_i S_i \times \prod_j C_j,$$

  where:

  1. *Z is a complex torus;*
  2. *Each S_i is a simply connected holomorphic symplectic manifold with $\dim H^2(S_i, \mathcal{O}_{S_i}) = 1$ (HyperKähler manifold);*
  3. *Each C_j is a simply connected Calabi-Yau manifold with $H^2(C_j, \mathcal{O}_{C_j}) = 0$.*

- For physicists: String theory is a branch of high-energy theoretical physics in which particles are modelled not as points but as 1-dimensional objects propagating in some background space-time M. For a number of physical reasons, string theorists suppose that the space we live in looks locally like $\mathbb{R}^4 \times X$, where $\mathbb{R}^4$ is the Minkowski space and X is a compact Riemannian manifold of dimension 6, with radius of order of 10^{-33}, the Planck length. Since the Planck length is so small, space appears to macroscopic observers to be 4-dimensional.
  Because of supersymmetry, X has to be a Calabi-Yau threefold.

1.3 Examples of Calabi-Yau threefolds

We recall the definition of normal bundle and the adjunction formula

Definition 2. Let $Y \subseteq X$ be a complex submanifold. The *normal bundle* of Y in X is the holomorphic vector bundle which is the cokernel of the injection $\mathcal{T}_Y \hookrightarrow \mathcal{T}_X|_Y$. Thus, there exists a short exact sequence of holomorphic vector bundles on Y, the *normal bundle sequence*:

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

Proposition 2. *Let Y be a complex submanifold of a complex manifold X. Then*

$$K_Y \cong \det\left(\mathcal{N}_{Y/X}\right) \otimes K_X|_Y.$$

1.3.1 Hypersurfaces in projective space

Let f be a homogeneous quintic polynomial in the variables $x_0, ..., x_4$. Then

$$X = \{x \in \mathbb{P}^4 \mid f(x) = 0\}$$

is a hypersurface in $\mathbb{P}^4$. Now we can compute K_X using the following result:

Proposition 3. *Let Y be a smooth hypersurface of a complex manifold X defined by a holomorphic section $s \in H^0(X, L)$, with L a holomorphic line bundle on X. Then*

$$\mathcal{N}_{Y/X} \cong L|_Y.$$

Since $f \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$, we obtain

$$N_{X/\,\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(5)|_X =: \mathcal{O}_X(5).$$

In particular,

$$K_X \cong \mathcal{O}_X(5) \otimes \mathcal{O}_X(-5) \cong \mathcal{O}_X.$$

We can compute some Hodge numbers using the following theorem

Theorem 2 (General version of the Weak Lefschetz theorem)**.** *Let $Y \subseteq X$ be a smooth complete intersection of ample line bundles and let $i : Y \hookrightarrow X$ be the canonical inclusion. Then th maps induced in singular cohomology*

$$H^k(i) : H^k(X, \mathbb{Z}) \longrightarrow H^k(Y, \mathbb{Z})$$

are:

- *bijective for $k < \dim Y$;*
- *injective for $k = \dim Y$.*

In our case, the Weak Lefschetz theorem implies:

- $\dim_{\mathbb{C}} H^0(X, \mathbb{C}) = \dim_{\mathbb{C}} H^0(\mathbb{P}^4, \mathbb{C}) = 1;$
- $\dim_{\mathbb{C}} H^1(X, \mathbb{C}) = \dim_{\mathbb{C}} H^1(\mathbb{P}^4, \mathbb{C}) = 0;$
- $\dim_{\mathbb{C}} H^2(X, \mathbb{C}) = \dim_{\mathbb{C}} H^2(\mathbb{P}^4, \mathbb{C}) = 1.$

So the Hodge diamond has the following form

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

We can calculate $h^{1,2}(X) = \dim_{\mathbb{C}} H^2(X, \Omega^1_X) = \dim_{\mathbb{C}} H^1(X, \mathcal{T}_X)$ as follows. Let us consider the following short exact sequences:

(i) The short exact sequence of vector bundles on X obtained by restricting the Euler sequence of $\mathbb{P}^4$ to X

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{T}_{\mathbb{P}^4}|_X \longrightarrow 0$$

(ii) The normal bundle sequence on X

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^4}|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^4} \cong \mathcal{O}_X(5) \longrightarrow 0$$

(iii) The short exact sequence of sheaves on $\mathbb{P}^4$ induced by the surjection $\mathcal{O}_{\mathbb{P}^4} \to i_*\mathcal{O}_X$, where $i : X \hookrightarrow \mathbb{P}^4$ is the canonical inclusion

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

The sequence (ii) gives us the long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{T}_X) \arrow[r] & H^0(\mathcal{T}_{\mathbb{P}^4}|_X) \arrow[r] & H^0(\mathcal{O}_X(5)) \arrow[dll, rounded corners] \\
 & H^1(\mathcal{T}_X) \arrow[r] & H^1(\mathcal{T}_{\mathbb{P}^4}|_X) \arrow[r] & H^1(\mathcal{O}_X(5)) \arrow[dll, rounded corners] \\
 & H^2(\mathcal{T}_X) \arrow[r] & H^2(\mathcal{T}_{\mathbb{P}^4}|_X) \arrow[r] & \cdots .
\end{tikzcd}$$

In order to compute $h^1(X, \mathcal{T}_X)$, we need $h^i(X, \mathcal{O}_X), h^i(X, \mathcal{O}_X(1)), h^i(X, \mathcal{O}_X(5))$. We proceed in the following way:

1. We tensor the sequence (i) with $\mathcal{O}_{\mathbb{P}^4}(d)$, with $d = 0, 1, 5$ and we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d) \longrightarrow i_*\mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_{\mathbb{P}^4}(d) \longrightarrow 0.$$

2. If $f : X \to Y$ is a morphism of ringed spaces, F is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then

$$f_*F \otimes_{\mathcal{O}_Y} \mathcal{E} \cong f_*(F \otimes_{\mathcal{O}_X} f^*\mathcal{E})$$

as $\mathcal{O}_Y$-modules.
So, if we take $F = \mathcal{O}_X$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^4}(d)$ then

$$i_*\mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_{\mathbb{P}^4}(d) \cong i_*i^*\mathcal{O}_{\mathbb{P}^4}(d),$$

which is the $i_*(\mathcal{O}_{\mathbb{P}^4}(d)|_X)$.

3. Moreover, let $f : X \to Y$ is an affine morphism of schemes, with X Noetherian, and let F be a quasi-coherent sheaf on X. Then

$$H^i(X, F) \cong H^i(Y, f_*F),$$

for every $i \geq 0$.

In this way, we get

$$\begin{aligned}
h^0(\mathcal{O}_X) &= 1\\
h^1(\mathcal{O}_X) &= 0\\
h^2(\mathcal{O}_X) &= 0\\
h^0(\mathcal{O}_X(1)) &= 5\\
h^1(\mathcal{O}_X(1)) &= 0\\
h^2(\mathcal{O}_X(1)) &= 0\\
h^0(\mathcal{O}_X(5)) &= 125\\
h^1(\mathcal{O}_X(5)) &= 0\\
h^2(\mathcal{O}_X(5)) &= 0.
\end{aligned}$$

Now we can compute

$$\begin{aligned}
h^0(\mathcal{T}_{\mathbb{P}^4}|_X) &= 24\\
h^1(\mathcal{T}_{\mathbb{P}^4}|_X) &= 0.
\end{aligned}$$

Moreover $h^0(\mathcal{T}_X) = h^3(\Omega^1_X) = h^{1,3} = 0$. Finally, substituting in the long exact sequence above, we get

$$h^{2,1} = 101.$$

Remark 2. If $X \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $n+1$, then X is a Calabi-Yau manifold of dimension $n-1$.

Remark 3. We can compute $h^{1,2}$ by counting the dimension of quintic hypersurfaces in $\mathbb{P}^4$. The dimension of the vector space of homogeneous polynomials of degree 5 in 5 variables is

$$\binom{4+5}{5} = 126.$$

However, proportional polynomials give the same hypersurface, and two polynomials related by an element of PGL(5) yield isomorphic hypersurfaces. This yields a moduli space of dimension $126-1-24 = 101$.

1.3.2 Complete intersections in projective space

Proposition 4. *If $X \subseteq \mathbb{P}^n$ is a complex manifold which is a complete intersection, i.e. $X = V(f_1, ..., f_r)$ with $f_i \in \mathbb{C}[x_0, ..., x_n]_{d_i}$ and* $\operatorname{codim}(X, \mathbb{P}^n) = r$, *then*

$$\mathcal{N}_{X/\mathbb{P}^n} \cong \bigoplus_{i=1}^{r} \mathcal{O}_X(d_i)$$

and

$$K_X \cong \mathcal{O}_X(-1-n+\sum_{i=1}^{r} d_i).$$

Therefore, the canonical line bundle K_X of a complete intersection[1] is trivial if and only if

$$\sum_{i=1}^{r} d_i = n+1.$$

Let us find all complete intersections of dimension 3 in $\mathbb{P}^n$. We suppose $d_i \geq 2$, for $i = 1, ..., r$. Indeed, if $d_i = 1$ for some i, then the complete intersection in contained in a hyperplane, so in a projective space with dimension smaller than n. Moreover, in order to have $\dim X = 3$, r must be equal to $n-3$. So

$$\sum_{i=1}^{n-3}(d_i - 1) + n - 3 = n + 1$$

and

$$n - 3 \leq \sum_{i=1}^{n-3}(d_i - 1) = 4 \Rightarrow n \leq 7.$$

Let us summarize what we found:

$(n = 4)$: $r = 1$ and X is a quintic hypersurface in $\mathbb{P}^4$.

$(n = 5)$: $r = 2$ and $d_1 + d_2 = 6$. So $X = V(f_2, g_4)$ or $X = V(f_3, g_3)$.

$(n = 6)$: $r = 3$ and $d_1 + d_2 + d_3 = 7$. So $X = V(f_2, g_2, h_3)$.

$(n = 7)$: $r = 4$ and $d_1 + d_2 + d_3 + d_4 = 8$. So $X = V(f_2, g_2, p_2, q_2)$.

As an example, we calculate the Hodge diamond of a complete intersection of type $(2, 4)$ in $\mathbb{P}^5$. By the weak Lefschetz theorem we obtain

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

We can compute $h^{2,1}$ using the Gauss-Bonnet-Chern theorem. The normal bundle sequence gives

$$c(X) \cdot c(\mathcal{N}_{X/\mathbb{P}^5}) = c(\mathcal{T}_{\mathbb{P}^5}|_X) = i^*c(\mathbb{P}^5) = 1 + 6h + 15h^2 + 20h^3,$$

where $h = i^*H$. Then

$$c(\mathcal{N}_{X/\mathbb{P}^5}) = c(\mathcal{O}_X(2)) \cdot c(\mathcal{O}_X(4)) = i^*c(\mathcal{O}_X(2)) \cdot i^*(\mathcal{O}_X(4)) = (1 + 2h^2)(1 + 4h^2).$$

After calculating the products, we get

[1] If $\dim X \geq 3$ and $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion, then $i^* : \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X)$ is an isomorphism. Compare the long exact sequences induced by the exponential sequence and, because the weak Lefschetz isomorphism is compatible with the Hodge decomposition, conclude.

- $c_1(X) + 6h = 6h \Rightarrow c_1(X) = 0$. We already knew that because $c_1(X) = -c_1(K_X)$;
- $c_2(X) + 8h^2 = 15h^2 \Rightarrow c_2(X) = 7h^2$;
- $c_3(X) + 6hc_2(X) = 20h^3 \Rightarrow c_3(X) = -22h^3$.

By Gauss-Bonnet-Chern formula

$$\chi(X) = \int_X c_3(X) = -22 \int_X h^3 = -22 \int_{\mathbb{P}^5} H^3 \wedge 2H \wedge 4H = -176 = 2(1 - h^{2,1}(X)).$$

So $h^{2,1}(X) = 89$. The Hodge diamond of a complete intersection of type $(2,4)$ in $\mathbb{P}^5$ is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 89 & & 89 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Remark 4.

- If X is the complete intersection of type $(3,3)$, then $h^{2,1}(X) = 73$;
- If X is the complete intersection of type $(2,2,3)$, then $h^{2,1}(X) = 73$;
- If X is the complete intersection of type $(2,2,2,2)$, then $h^{2,1}(X) = 65$.

1.3.3 Complete intersections in a product of projective spaces

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and let $\pi_i : Y \to \mathbb{P}^{n_i}$ be the i-th projection. We define the line bundles

$$\mathcal{O}_Y(1_i) := \pi_i \mathcal{O}_{\mathbb{P}^{n_i}}(1)$$

and

$$\mathcal{O}_Y(a_1, \dots, a_n) := \bigotimes_{i=1}^r \mathcal{O}_Y(1_i)^{\otimes a_i}.$$

Remark 5. Every line bundle on Y has this form.

Proposition 5.

$$K_Y \cong \mathcal{O}_Y(-n_1 - 1, \dots, -n_r - 1).$$

Moreover

Proposition 6. *Let $X = V(f_1, \dots, f_r) \subseteq Y$ be a smooth complete intersection, with f_i a multihomogeneous of multidegree $(a_1^i, \dots, a_r^i)$. Then*

$$\mathcal{N}_{X/Y} \cong \bigoplus_{i=1}^r \mathcal{O}_X(a_1^i, \dots, a_r^i).$$

Then, using the adjunction formula, we obtain some Calabi-Yau threefolds as complete intersections in a product of projective spaces.

1.3.4 Calabi-Yau manifolds from singular varieties

If X is a normal singular variety, then the singular locus of X has codimension ≥ 2 in X. Let $i : X_{ns} \hookrightarrow X$ be the inclusion. If $i_* K_{X_{ns}}$ is a locally free $\mathcal{O}_X$-module of rank 1, we say that X is Gorestein and we then define K_X as $i_* K_{X_{ns}}$.
Let X be Gorestein. A resolution of singularity $f : Y \to X$, i.e. a birational morphism from a non singular variety to X, is *crepant* if

$$f^* K_X \cong K_Y.$$

Remark 6. In general, a crepant resolution of singularity may not exist.

We consider $\mathbb{P}^2 = V(x_0, x_1) \subseteq \mathbb{P}^4$ and a quintic hypersurface X containing $\mathbb{P}^2$, i.e. any quintic of the form

$$x_0 f + x_1 g$$

with $f, g \in \mathbb{C}[x_0, \ldots, x_4]$ homogeneous polynomials of degree 4. It is straightforward to verify that X is singular in the 16 points given by the equation

$$x_0 = x_1 = f = g = 0.$$

The adjunction formula still holds for singular hypersurfaces of degree d. So $K_X \cong \mathcal{O}_X$. In this case every resolution of singularity $f : Y \to X$, with $K_Y \cong \mathcal{O}_Y$, is crepant.
Let $Z = V(x_0 v - x_1 u) \subseteq \mathbb{P}^1_{[u:v]} \times \mathbb{P}^4$ be the blow up of $\mathbb{P}^4$ along $\mathbb{P}^2$. It is straightforward to verify that the equation of the strict transform $\tilde{X}$ of X is given by

$$x_0 v - x_1 u = uf + vg = 0.$$

If f, g are generic quartic polynomials, then $\tilde{X}$ is a smooth threefold in $\mathbb{P}^1 \times \mathbb{P}^4$ and we can calculate its canonical bundle by the adjunction formula. Moreover, we can recover its Hodge diamond using the weak Lefschetz theorem and the same short exact sequences[2] used for complete intersections in projective space. In particular we obtain

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Therefore $\pi : \tilde{X} \to X$ is a crepant resolution of singularity.

[2] The normal bundle sequence, the Euler sequence for products of projective space and the Koszul complex.

Chapter 2

Mirror symmetry

In string theory, each Calabi-Yau threefold is equipped with a super conformal field theory, which a is a complicated mathematical object. However, two non isomorphic Calabi-Yau threefolds X and $\hat{X}$ may have the same SCFD, and in that case there are powerful relations between invariants of X and $\hat{X}$ (for example $h^{1,1}(X) = h^{2,1}(\hat{X})$ and viceversa). This is the idea behind *Mirror Symmetry* of Calabi-Yau threefolds.

In the beginning (the 1980's), Mirror Symmetry seemed mathematically completely mysterious. But there are now two complementary conjectural theories, due to Kontsevich and Strominger-Yau-Zaslow, which explain Mirror Symmetry in a fairly mathematical way. Probably both are true, at some level. The first proposal was due to Kontsevich in 1994. This states that for mirror Calabi-Yau threefolds X and $\hat{X}$, the derived category $D^b(X)$ of coherent sheaves on X is equivalent to the derived category $D^b(\mathrm{Fuk}(\hat{X}))$ of the Fukaya category of $\hat{X}$, and viceversa. Basically, $D^b(X)$ has to do with X as a complex manifold, and $D^b(\mathrm{Fuk}(\hat{X}))$ with $\hat{X}$ as a symplectic manifold, and its Lagrangian submanifolds. The second proposal, due to Strominger, Yau and Zaslow in 1996, is known as the *SYZ Conjecture*. Here is an attempt to state it.

The SYZ Conjecture *Suppose X and $\hat{X}$ are mirror Calabi–Yau threefolds. Then (under some additional conditions) there should exist a compact topological 3-manifold B and surjective, continuous maps $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *There exists a dense open set $B_0 \subset B$, such that for each $b \in B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are nonsingular special Lagrangian 3-tori T^3 in X and $\hat{X}$. Furthermore, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are dual to one another.*

(ii) *For each $b \in \Delta = B \setminus B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are expected to be singular special Lagrangian 3-folds in X and $\hat{X}$.*

We call f and $\hat{f}$ *special Lagrangian fibrations*, and the set of singular fibres Δ is called the *discriminant*.

In particular, there is a relation between the moduli space of complex structure on X and the complexified moduli space of Kähler classes of the mirror $\hat{X}$.

Chapter 3

Appendix on Chern classes

There are different ways to define the Chern classes of a vector bundle: we will follow the axiomatic approach by Grothendieck.

Definition 3. Let X be a topological space and $\pi : E \to X$ a complex vector bundle. We define the k-th *Chern class* $c_k(E) \in H^{2k}(X, \mathbb{Z})$ and the *total Chern class* $c(E) = \sum c_i(E) \in H^*(X, \mathbb{Z})$ as the unique classes in the singular cohomology of X satisfying:

1. $c_0(E) = 1$, for every vector bundle E;
2. If $f : X \to Y$ is a continuous map, then $c_k(f^*E) = H^{2k}(f)(c_k(E))$;
3. If $0 \to E \to F \to G \to 0$ is a short exact sequence of vector bundles, then $c(F) = c(E) \cdot c(G)$;
4. $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = H$, where H is the Poincaré dual of a hyperplane in $\mathbb{P}^n$.

Definition 4. Let M be a compact oriented manifold and let $Z \subseteq X$ a closed oriented submanifold of codimension k. Then the *Poincaré dual class* of Z is the unique class $\eta_Z \in H^k(X, \mathbb{R})$ such that

$$\int_Z \alpha|_Z = \int_X \alpha \wedge \eta_Z,$$

for each $\alpha \in H^{n-k}(X)$

Proposition 7. *The Poincaré dual of a smooth complete intersection $X = V(f_{d_1}, ..., f_{d_r}) \subseteq \mathbb{P}^n$, where f_{d_i} is a homogeneous polynomial of degree d_i in $n+1$ variables, is*

$$\eta_X = d_1 H \wedge \cdots \wedge d_r H,$$

with H the Poincaré dual of a hyperplane in $\mathbb{P}^n$.

Lemma 1. *From the axiomatic properties of the Chern classes we can deduce:*

1. *If $E \cong \mathcal{O}_X$, then $c(E) = 1$;*
2. *$c_k(E) = 0$ for $k > \min\{\dim X, \operatorname{rk} E\}$;*
3. *$c_1(E \otimes F) = \operatorname{rk} E c_1(F) + \operatorname{rk} F c_1(E)$;*

4. $c_k(E^\vee) = (-1)^k c_k(E)$;

5. $c_1(E) = c_1(\det E)$.

Definition 5. Let X be a complex manifold. Then $c(X) := c(\mathcal{T}_X)$.

Proposition 8.

$$c(\mathbb{P}^n) = \sum_{k=0}^{n} \binom{n+1}{k} H^k.$$

Proposition 9.

$$\int_{\mathbb{P}^n} H^n = 1.$$

Theorem 3 (Gauss-Bonnet-Chern theorem)**.** *Let X be a compact complex Kähler manifold of dimension n. Then*

$$\chi(X) = \int_X c_n(X).$$